FOURTH ORDER COMPACT METHOD FOR ONE DIMENSIONAL HOMOGENEOUS DAMPED WAVE EQUATION

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Abstract: Finite Difference Method (FDM) and Fourth Order Compact Method (FOCM) are presented in this paper for the solutions of the well known one dimensional Homogeneous Damped Wave Equation. First we develop a general scheme for FDM and FOCM for the Damped Wave Equation and then its validity is checked through applications. The results obtained by the FDM and FOCM are compared with the exact solutions for these applications. Results have obtained numerically and graphically. We use FORTRAN 90 for the calculation of the numerical results and MS Office for graphical comparison.

Key words: Finite Difference Method, Fourth Order Compact Method, Damped Wave Equation.

INTRODUCTION

The hyperbolic PDE's model the vibrations of structures like buildings, beams and machine etc., and make the base for fundamental equations of atomic physics. Fourth Order Compact method uses only three grid points for approximating the solution as compared to the standard finite method which uses five grid points to obtain the same kind of accuracy. ORSZAG in 1974 presented this Compact formula and CIMENT and LEVENTHAL in 1978 applied this method on hyperbolic problems.

Consider the second order one dimensional homogenous damped wave equation

$$a\frac{\partial^2 u(x,t)}{\partial t^2}+b\,\frac{\partial u\,(x,t)}{\partial t} = c^2\,\frac{\partial^2 u(x,t)}{\partial x^2} \qquad (1)$$

Initial condition

$$u(x,0) = f(x) \qquad (2)$$

$$u_t(x,0) = \frac{\partial u\,(x,0)}{\partial t} = g(x) \qquad (3)$$

Boundary condition

$$u(0,t) = 0 \qquad (4)$$

$$u(l,t) = 0 \qquad (5)$$

for $0 \leq x \leq l\,, 0 \leq t \leq N$

Equation (1) is the second order Damped Wave Equation with constant coefficients. In equation (1), x is displacement and t is time.

Equations like damped wave equation arise in the study of viscoelastic theory (Stig Larsson, 1989). The process of energy dissipation is generally referred to as damping. Damping in general, has the effect of reducing the amplitude of vibration and therefore, it is desirable to have some amount of damping in order to achieve stability (Robert Earl, 2004). A suitable subset of the domain is supported the damping term, therefore as time goes to infinity the energy of solutions of the damped wave equation decays exponentially to zero (Arnaus and Admire, 2007). It has been indicated by several mathematicians show as $t \to \infty$ the damped wave equation has the diffusive structure. (Woodhouse, 2004) used exponential functions to curve-fit the decay profiles of harmonics from a plucked guitar string in order to quantify the amount of damping for each harmonic. (Hasselman, 1972) outlined a method whereby a proportional, viscous damping matrix could be reconstructed from modal frequencies and knowledge of the mass and stiffness matrices.

Finite Difference Scheme: To develop the finite difference scheme for equation (1), we choose a digit m and the time step t from 0 to ∞, the engage points (x_i, t_n) are

$$x_i = i\Delta x = ih \quad for\ i = 0,1,2,m.$$

$$t_n = n\Delta t = nk \quad for\ n = 0,1,2,\ldots$$

At any interior engage point (x_i, t_n) , the Hyperbolic Homogenous Damped Wave equation (1), becomes

$$a\frac{\partial^2 u(x_i,t_n)}{\partial t^2}+b\,\frac{\partial u\,(x_i,t_n)}{\partial t} = c^2\,\frac{\partial^2 u(x_i,t_n)}{\partial x^2} \qquad (6)$$

The scheme is developed using the central difference approximation for the second order partial derivatives and central difference approximation for the first order partial derivative. Thus (6) becomes

$$\frac{a}{k^2}(u_i^{n+1} - 2u_i^n + u_i^{n-1}) + \frac{b}{2k}(u_i^{n+1} - u_i^{n-1}) = \frac{c^2}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$\left(\frac{a}{k^2} + \frac{b}{2k}\right)u_i^{n+1} + \left(\frac{a}{k^2} - \frac{b}{2k}\right)u_i^{n-1} = \frac{c^2}{h^2}(u_{i+1} + u_{i-1}^n) + \frac{2a}{k^2}u_i^n - \frac{2c^2}{h^2}u_i^n$$

$$\left(\frac{a}{k^2} + \frac{b}{2k}\right)u_i^{n+1} = \left(\frac{b}{2k} - \frac{a}{k^2}\right)u_i^{n-1} + \frac{c^2}{h^2}(u_{i+1}^n + u_{i-1}^n) + 2\left(\frac{a}{k^2} - \frac{c^2}{h^2}\right)u_i^n$$

$$\lambda_1 u_i^{n+1} = \lambda_2 u_i^{n-1} + \lambda_3 u_{i+1}^n + \lambda_3 u_{i-1}^n + \lambda_4 u_i^n$$

where $\left(\frac{a}{k^2}+\frac{b}{2k}\right)=\lambda_1$, $\left(\frac{b}{2k}-\frac{a}{k^2}\right)=\lambda_2$, $\frac{c^2}{h^2}=\lambda_3$, $2\left(\frac{a}{k^2}-\frac{c^2}{h^2}\right)=\lambda_4$

or $u_i^{n+1}=\frac{\lambda_2}{\lambda_1}u_i^{n-1}+\frac{\lambda_3}{\lambda_1}u_{i+1}^n+\frac{\lambda_3}{\lambda_1}u_{i-1}^n+\frac{\lambda_4}{\lambda_1}u_i^n$ (7)

This equation valid for each $i=1,2,\dots,(m-1)$

The B.C. give

$$u_0^n=u_m^n=0 \qquad (8)$$

for each $n=1,2,\dots$

and the I.C. implies that

$$u_i^0=f(x_i) \qquad (9)$$

for $i=1,2,\dots,(m-1)$.

Equation (7) and (8) mean that the $(n+1)$th occasion steps needs values from the nth and $(n-1)$th occasion steps. This creates a minor starting problem since values of $n=1$ which is needed to compute u_i^2 in (7), must be obtained from the I.C.

$$u_t|_i^0=g(x_i)\,,0\le x\le l$$

An improved approximation $u_t|_i^0$ can be obtained slightly easily, mainly when the second derivative of $'f'$ at x_i can be firmed.

Consider the Taylor series

$$u_i^{n+1}=u_i^n+ku_t|_i^n+\frac{k^2}{2}u_{tt}|_i^n+O(k^3)$$

for $n=0$, we have

$$\frac{u_i^1-u_i^0}{k}=u_t|_i^0+\frac{k}{2}u_{tt}|_i^0+O(k^2) \qquad (10)$$

Suppose the homogenous damped wave equation also holds on the initial condition. i.e., by using (2).

$$u_{tt}|_i^0=\frac{c^2}{a}f_{xx}(x_i)-\frac{b}{a}u_t|_i^0$$

Substituting this value in equation (10), we get

$$\frac{u_i^1-u_i^0}{k}=u_t|_i^0+\frac{kc^2}{2a}f_{xx}(x_i)-\frac{kb}{2a}u_t|_i^0+O(k^2)$$

$u_i^1=\frac{k^2c^2}{2a}f_{xx}(x_i)+\left(k-\frac{k^2b}{2a}\right)g(x_i)+u_i^0$, by using (3)

This is estimation with local truncation fault $O(k^3)$ for each $i=1,2,\dots,m-1$. Now by using the central difference approximation for f_{xx} the last result becomes,

$$u_i^1=\frac{\lambda^2}{2a}\left(f(x_{i+1})+f(x_{i-1})\right)+\left(k-\frac{k^2b}{2a}\right)g(x_i)+\left(1-\frac{\lambda^2}{a}\right)f(x_i) \quad (11)$$

for $i=1,2,\dots,(m-1)$, where $\lambda^2=\frac{k^2c^2}{h^2}$

Compact Scheme for Damped Wave Equation: Now we derive the Fourth Order Compact Scheme for the second order linear homogeneous Damped Wave equation(1) with $a>0, b>0, c^2>0, f(x)$ *and* $g(x)$ are given functions.

Let

$$u_x(x,t)=F \qquad (12)$$

$$u_{xx}(x,t)=S \text{ or } u_{xx}(x,t)=F_x=S$$

Integrating both sides of equation (12) from $i-1$ to $i+1$ and estimating this integral by Simpson's Rule, we acquire

$$u_{i+1}^n=u_{i-1}^n+\frac{h}{3}(F_{i-1}^n+4F_i^n+F_{i+1}^n)+O(h^5)$$

Thus to fourth order, we get

$$F_{i-1}^n+4F_i^n+F_{i+1}^n+\frac{3}{h}(u_{i+1}^n-u_{i-1}^n)=0 \qquad (13)$$

Equation (13) is the first difference equation. To find second equation, we start by solving (1) at the midpoint i, then equation (1) becomes

$$au_{tt}|_i^n+bu_t|_i^n=c^2S|_i^n \qquad (14)$$

Now we need the expression for $S|_i^n$. Express u_{i+1}^n and u_{i-1}^n in the Taylor's series regarding the point (i,n) and adding the result we get

$$u_{i+1}^n+u_{i-1}^n=2u_i^n+h^2S|_i^n+\frac{h^4}{12}u_{xxxx}|_i^n+O(h^6) \qquad (15)$$

Similarly expanding F_{i+1}^n and F_{i-1}^n by using the Taylor's series and subtracting the result we get

$$F_{i+1}^n-F_{i-1}^n=2hS|_i^n+\frac{h^3}{3}u_{xxxx}|_i^n+O(h^5) \qquad (16)$$

We now remove the $u_{xxxx}|_i^n$ from equations (15) and (16) and solve for $S|_i^n$, we get

$$S|_i^n=\frac{2}{h^2}(u_{i+1}^n+u_{i-1}^n-2u_i^n)-\frac{1}{2h}(F_{i+1}^n-F_{i-1}^n)+O(h^4) \qquad (17)$$

By a similar procedure we get the following expressions for $S|_{i-1}^n$ and $S|_{i+1}^n$

$S|_{i-1}^n=\frac{1}{2h^2}(-23u_{i-1}^n+16u_i^n+7u_{i+1}^n)-\frac{1}{h}(6F_{i-1}^n+$ *8Fin+ Fi+1n+Oh4* (18)

and

$S|_{i+1}^n=\frac{1}{2h^2}(7u_{i-1}^n+16u_i^n-23u_{i+1}^n)+\frac{1}{h}(F_{i-1}^n+8F_i^n+$ *6Fi+1n+Oh4* (19)

From equation (17) substitute the expression for $S|_i^n$ into equation (14) and on simplifying, we get

$$au_{tt}|_i^n+bu_t|_i^n=\frac{2c^2}{h^2}(u_{i+1}^n+u_{i-1}^n)-\frac{4c^2}{h^2}u_i^n-\frac{c^2}{2h}(F_{i+1}^n-F_{i-1}^n) \qquad (20)$$

Consider the left B.C. at $x=0$ and denotes the points $x=0,h,2h$ by 0,1,2. The first difference equation we obtain from the B.C. is

$$u_0^n=0 \qquad (21)$$

The second equation can be obtained by using equation (14) and equation (17), equation (18) and equation (13) at $i=1$, so we get five equations. Then on eliminating u_2^n, S_0^n, S_1^n *and* F_2^n from these five equations, we get the second difference equation valid at $x=0$. Thus we

$$a(u_{tt}|_1^n-u_{tt}|_0^n)+b(u_t|_1^n-u_t|_0^n)=\frac{12c^2}{h^2}u_0^n-\frac{12c^2}{h^2}u_1^n+\frac{6c^2}{h}F_0^n+\frac{6c^2}{h}F_1^n \qquad (22)$$

Equation (22) is the second difference equation applicable at $x=0$. Similarly, we can obtain the following difference equation for u and F at $x=m$ i.e. at the right boundary point.

$$u_m^n=0 \qquad (23)$$

$a(u_{tt}|_m^n - u_{tt}|_{m-1}^n) + b(u_t|_m^n - u_t|_{m-1}^n) = \frac{12c^2}{h^2}u_{i-1}^n - \frac{12c^2}{h^2}u_m^n + \frac{6c^2}{h}F_{m-1}^n + \frac{6c^2}{h}F_m^n$ (24)

Hence we have two difference equations corresponding to each point. Now by using central difference approximation for $u_{tt}|_i^n$ and central difference approximation for $u_t|_i^n$ we have

$$u_{tt}|_i^n = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{k^2} + O(k^2)$$

$$u_{tt}|_{i-1}^n = \frac{u_{i-1}^{n+1} - 2u_{i-1}^n + u_{i-1}^{n-1}}{k^2} + O(k^2)$$

and

$$u_t|_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2k} + O(k^2)$$

$$u_t|_{i-1}^n = \frac{u_{i-1}^{n+1} - u_{i-1}^{n-1}}{2k} + O(k^2)$$

then

$$u_{tt}|_i^n - u_{tt}|_{i-1}^n = \frac{(u_i^{n+1} + u_i^{n-1}) - 2(u_i^n - u_{i-1}^n) - (u_{i-1}^{n-1} + u_{i-1}^{n+1})}{k^2}$$

and

$$u_t|_i^n - u_t|_{i-1}^n = \frac{(u_i^{n+1} - u_i^{n-1} + u_{i-1}^{n-1} - u_{i-1}^{n+1})}{2k}$$

Then by using the last results we have from equations (21), (22), (13), (20), (24) and (23) the ultimate Fourth Order Compact Scheme as under

$$u_0^n = 0$$

$$\left(\frac{a}{k^2} + \frac{b}{2k}\right)u_1^{n+1} - \frac{6c^2}{h}F_0^n - \frac{6c^2}{h}F_1^n = \left(\frac{2a}{k^2} - \frac{12c^2}{h^2}\right)u_1^n + \left(\frac{b}{2k} - \frac{a}{k^2}\right)u_1^{n-1}$$

$$F_{i-1}^n + 4F_i^n + F_{i+1}^n + \frac{3}{h}(u_{i+1}^n - u_{i-1}^n) = 0$$

$$\left(\frac{a}{k^2} + \frac{b}{2k}\right)u_i^{n+1} + \frac{c^2}{2h}F_{i+1}^n - \frac{c^2}{2h}F_{i-1}^n = \frac{2c^2}{h^2}(u_{i+1}^n - u_{i-1}^n) + \left(\frac{2a}{k^2} - \frac{4c^2}{h^2}\right)u_i^n + \left(\frac{b}{2k} - \frac{a}{k^2}\right)u_i^{n-1}$$

$$\left(\frac{a}{k^2} + \frac{b}{2k}\right)u_{m-1}^{n+1} + \frac{6c^2}{h}F_{m-1}^n + \frac{6c^2}{h}F_m^n = \left(\frac{2a}{k^2} - \frac{12c^2}{h^2}\right)u_{m-1}^n + \left(\frac{b}{2k} - \frac{a}{k^2}\right)u_{m-1}^{n-1}$$

$$u_m^n = 0$$

Here $F_0^n, F_1^n, F_{i-1}^n, F_i^n, F_{i+1}^n, u_i^{n+1}, F_m^n, F_{m-1}^n$ are unknowns and

$$u_0^{n+1} = u_m^{n+1} = u_0^n = u_m^n = u_0^{n-1} = u_m^{n-1} = 0$$

Now for finding u_i^1 for the next time level, we use the initial condition

$$u_t|_i^0 = u_t(x,0) = g(x_i), \qquad 0 \le x \le l$$

Which can be approximated into the form by using Taylor's series and finite difference as under

$$u_i^1 = \frac{\lambda^2}{2a}\left(f(x_{i+1}) + f(x_{i-1})\right) + \left(k - \frac{k^2 b}{2a}\right)g(x_i) + \left(1 - \frac{\lambda^2}{a}\right)f(x_i) \tag{25}$$

Which can be used to find u_i^1 for each $i = 1,2,\dots,(m-1)$, where $\lambda^2 = \frac{k^2c^2}{h^2}$

Application 1

Consider the homogeneous damped wave equation $u_{tt} + 3u_t = 2u_{xx}$ in $0 \le x \le \pi$.

B.C. $u(0,t) = u(\pi,t) = 0$

I.C. $u(x,0) = \sin x,\ 0 \le x \le \pi \text{ and } 0 \le t \le \pi$

$u_t(x,0) = -2\sin x,\ 0 \le x \le \pi \text{ and } 0 \le t \le \pi$

The exact solution $u(x,t) = e^{-2t}\sin x$.

Table 1: Contrast of the numerical consequences of FDM at t=0.2

x_i	FDM	Exact	Error
.00000000	.00000000	.00000000	.00000000
.31415926	.20795413	.20714028	.00081384
.62831853	.39555226	.39400423	.00154802
.94247779	.54443098	.54230030	.00213067
1.25663706	.64001700	.63751224	.00250476
1.57079632	.67295370	.67032004	.00263366
1.88495559	.64001700	.63751224	.00250476
2.19911485	.54443098	.54230030	.00213067
2.51327412	.39555226	.39400423	.00154802
2.82743338	.20795413	.20714028	.00081384
3.14159265	.00000000	.00000000	.00000000

Table 2: Contrast of the numerical consequences of FOCM at t=0.2

x_i	FOCM	Exact	Error
.00000000	.00000000	.00000000	.00000000
.31415926	.20792216	.20714028	.00078188
.62831853	.39548216	.39400423	.00147793
.94247779	.54433639	.54230030	.00203608
1.25663706	.63990536	.63751224	.00239311
1.57079632	.67283645	.67032004	.00251640
1.88495559	.63990536	.63751224	.00239311
2.19911485	.54433639	.54230030	.00203608
2.51327412	.39548216	.39400423	.00147793
2.82743338	.20792216	.20714028	.00078188
3.14159265	.00000000	.00000000	.00000000

For graph of table 1 and table 2 observe Figure 1.

Application 2

Consider the damped wave equation $u_{tt} + 2u_t = u_{xx}$ in $0 \le x \le \pi$.

B.C. $u(0,t) = u(\pi,t) = 0$

I.C. $u(x,0) = \sin x,\ 0 \le x \le \pi \ and \ 0 \le t \le 1$ $\quad u_t(x,0) = -\sin x, 0 \le x \le \pi \ and \ 0 \le t \le 1$

The exact solution $u(x,t) = e^{-t} \sin x$.

Table 3: Contrast of the numerical consequences of FDM and at t=0.2

x_i	FDM	Exact	Error
.00000000	.00000000	.00000000	.00000000
.31415926	.24774239	.25300171	.00525932
.62831853	.47172494	.48123786	.00951292
.94247779	.64859801	.66236709	.01376900
1.25663706	.76247269	.77865921	.01618652
1.57079632	.80171123	.81873075	.01701952
1.88495559	.76247269	.77865921	.01618652
2.19911485	.64859801	.66236709	.01376900
2.51327412	.47172494	.48123786	.00951292
2.82743338	.24774239	.25300171	.00525932
3.14159265	.00000000	.00000000	.00000000

Table 4: Contrast of the numerical consequences of FOCM at t=0.2

x_i	FOCM	Exact	Error
.00000000	.00000000	.00000000	.00000000
.31415926	.25312428	.25300171	.00012256
.62831853	.48146563	.48123786	.00022777
.94247779	.66268168	.66236709	.00031458
1.25663706	.77902877	.77865921	.00036955
1.57079632	.81911940	.81873075	.00038865
1.88495559	.77902877	.77865921	.00036955
2.19911485	.66268168	.66236709	.00031458
2.51327412	.48146563	.48123786	.00022777
2.82743338	.25312428	.25300171	.00012256
3.14159265	.00000000	.00000000	.00000000

For graph of table 3 and table 4 observe Figure 2.

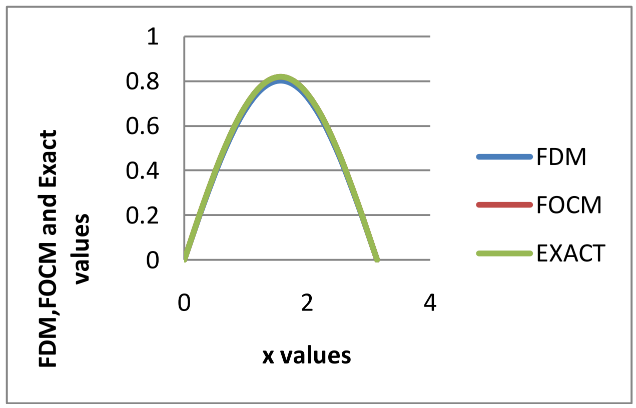

Figure 1: Contrast of FDM, FOCM and Exact values

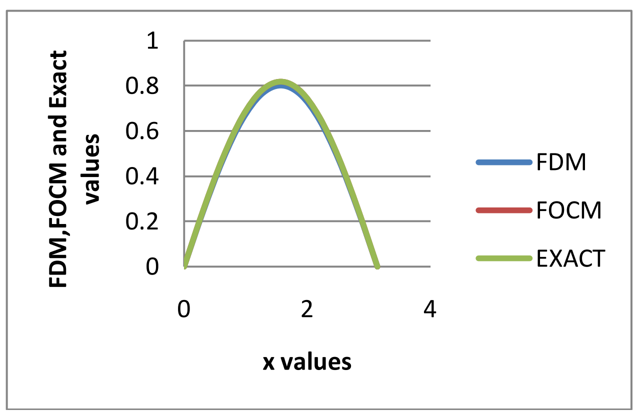

Figure 2: Contrast of FDM, FOCM and Exact values

Conclusion: The numerical solutions so computed by Finite Difference Method and Fourth Order Compact Method were compared with the exact numeric solution, and it is found that the results obtained by Fourth Order

Compact Method are more close to the exact solution than the solutions obtained by Finite Difference Method. FOCM is found to be a powerful tool for the numerical solution of PDE'S as well as for ODE'S. FOCM gives same accuracy with only three grid points as the other compact methods use five grid points for the same accuracy.

REFERENCES

Bellout. H. and A. Friedman. *Blow up estimates for a nonlinear hyperbolic heat equation*, Siam J: 354-366 (1989).

Hasselman, T. K. *Method for Constructing a Full Modal Damping Matrix from Experimental Measurements*, AIAA (10):526-527 (1972).

Munch. A and F. P. Admir. *uniform stabilization of a viscous numerical approximation for a locally damped wave equation*, Esaim (13): 265-293 (2007).

Muhammad. O. A. *An Exploration of Compact Finite Difference Methods for the Numerical Solution of PDE*, Ph D Thesis University of Western Ontario, Canada (1997).

Robert. E. P. *Numerical investigation of a damped wave equation with distributed and boundary energy dissipation*, M.Sc thesis, Texas Tech University America, August (2004).

Woodhouse, J. P. G Transients. *Comparison of Measurements and Synthesis*, Acta Acustica united with Acustica 90(5): 945-965 (2004).