VIBRATION CHARACTERISTICS OF CYLINDRICAL SHELLS WITH FGM MIDDLE LAYER BASED ON ELASTIC FOUNDATIONS UNDER VARIOUS BOUNDARY CONDITIONS

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ABSTRACT: In the current analysis vibration characteristics of a cylindrical shell composed of three layers are examined. This configuration is framed by three layers of different materials in thickness direction such that the inner and outer layers are of isotropic nature and functionally graded material is used for the middle layer. The shell is supported on Winkler and Pasternak foundations. Love shell equations are considered to study the vibration problem. The Winkler and Pasternak foundations are combined with the shell dynamical equations in the transverse direction. The present shell problem is solved by using wave propagation approach. A few comparisons of shell frequencies are done to verify the validity and accuracy of the present technique.

Key words: Three layered cylindrical shells, functionally graded material, Winkler and Pasternak elastic foundations, wave propagation approach, boundary conditions

INTRODUCTION

The problem of shell vibration was first studied by Sophie Germanie in 1821. (Rayleigh, 1884) and (Love, 1888) then analyzed it in last quarter of nineteenth century. Arnold and Warburton (1948, 1953) did some seminal and pioneering study on vibration characteristic of cylindrical shells in which the complete boundary value problem of a finite cylindrical shell was studied in detail. Sewall and Naumann (1968) analyzed the experimental and analytical vibration characteristics of stiffened and unstiffened cylindrical shells. (Paliwal *et al.* 1996) investigated the vibrations of a thin circular cylindrical shell attached with elastic foundations. Membrane theory was employed and response of the elastic foundations was represented by Winkler and Pasternak models. (Loy *et al.* 1998) investigated the vibrations of functionally graded material cylindrical shells, made up of FG material composed of stainless steel and nickel. The purpose of work was to examine natural frequencies, influence of the constituent volume fractions and effects of configurations of constituent materials on their frequencies. (Zhang *et al.* 2001a) analyzed the vibrations of cylindrical shells employing wave propagation approach. Comparison of numerical results obtained by using the wave propagation approach and numerical finite element method were executed. (Bing *et al.* 2005) analyzed natural frequencies of thin-walled circular cylindrical shells under several end conditions and used Donnell's thin shell theory and basic equations based on the wave propagation method. Mode shapes were drawn to describe the circumferential mode number *n* and axial mode number *m*, and these natural frequencies are computed numerically and compared with those of finite element method to confirm the reliability of theoretical solutions. (Goncalves *et al*. 2006) used a qualitatively accurate low dimensional model to investigate the non-linear motion behaviour of shallow cylindrical shells under axial loading. The dynamical version of the Donnell non-linear shallow shell equations were discretized by the Galerkin approach.

Li and Batra (2006) analyzed buckling of simply-supported three layer circular cylindrical shell under axial compressive load. Wang and Lin (2006) presented the formulation of motion equations for a symmetric cross-ply laminated cylindrical shell with a circumferential stiffener. Two types of the circumferential stiffeners were taken: outer ring and inner ring. (Zhang *et al*. 2006) presented the formulation and numerical analysis of circular cylindrical shells by the local adaptive differential quardrature method, which was applied to both localized interpolating basis functions and exterior grid points for boundary treatments. (Pellicano, 2007) presented a method for analyzing linear and non-linear vibrations of circular cylindrical shells based on different end conditions. Simply supported and clamped-clamped conditions were analyzed. (Arshad *et al.* 2007) derived the frequency equation in the form of eigen-value problem by employing Rayleigh-Ritz method. Love's thin shell theory was used for strain-displacement and curvature-displacement relation. They studied the natural frequency for simply-supported boundary conditions and compared the results with those mentioned in the literature to check the validity of the approach. (Iqbal *et* *al.* 2009) applied wave propagation approach to analyze vibrations of functionally graded material circular cylindrical shells. This methodology was very easy to apply. Axial model dependence was carried out by exponential functions. (Shah *et al*. 2009b) studied the effects of volume fraction law on the vibration frequencies of thin FG cylindrical shells. Material parameters in shell radial direction were ranked according to the exponential law. Love's thin shell theory was used in order to derive the expressions for the strain-displacement and curvature-displacement relationships. To get the shell equations of motion, Rayleigh-Ritz approach was applied. (Arshad *et al*. 2011) studied an analysis on vibrations of bi-layered cylindrical shell made of two layers which were functionally graded. The thickness of the shell layers was considered to be equal and constant. (Naeem *et al*. 2010) studied the vibration frequency characteristics of functionally graded cylindrical shells using the generalized differential quadrature method. The method was founded on the approximation of the derivatives of the unknown functions involved in differential equations at the mesh points of the solution domain. It was a sophisticated technique that gives accurate and robust results. A number of comparisons were done to check the effectiveness, robustness and accuracy of the presented method.

Formulation: A thin-walled cylindrical shell is considered here with the geometrical parameters: length *L*, thickness *h* and mean radius *R*. The orthogonal coordinate system (x, φ, z) is taken to be at the surface of the shell where *x*, φ and *z* represent the axial, circumferential and radial coordinates respectively. Young's modulus *E*, the Poisson ratio *v* and the mass density *ρ* are the shell material parameters. The axial, circumferential and radial displacement deformations are denoted by $u(x,\varphi,t)$, $v(x,\varphi,t)$ and $w(x,\varphi,t)$ respectively with regard to the shell middle surface. Strain energy *U* for a thin vibrating cylindrical shell is written as:

$$U = \frac{1}{2}\int_0^L \int_0^{2\pi} \{\epsilon\}^T [K]\{\epsilon\} R\, d\varphi\, dx \qquad (1)$$

where $\{\epsilon\}^T, [K]$ and $\{\epsilon\}$ are defined as:

$$\{\epsilon\}^T = \{e_1, e_2, \gamma, \kappa_1, \kappa_2, 2\tau\} \qquad (2)$$

$$[K] = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \qquad (3)$$

where A, B and D are sub-matrices of extensional, coupling and bending stiffness and for plane stress condition, they are given as:

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad [B] = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}, \quad [D] = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$$

where A_{ij}, B_{ij} and $D_{ij}(i, j = 1,2 \text{ and } 6)$ stand for extensional, coupling and bending stiffness respectively and defined as:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{+h/2} Q_{ij}(1, z, z^2)dz \qquad (4)$$

Kinetic energy T of a thin vibrating cylindrical shell attains the following form:

$$T = \int_0^L \int_0^{2\pi} \rho_t \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] R\, d\varphi\, dx \qquad (5)$$

where *t* symbolizes the time variable and ρ_t denotes the mass density per unit length and is defined as:

$$\rho_t = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho\, dz \qquad (6)$$

Love's thin shell theory is applied for strain-displacement and curvature-displacement relationships. The Lagrangian energy functional (Π) which is the difference of strain and kinetic energies, is written as:

$$\Pi = T - U \qquad (7)$$

Hamilton's principle is utilized to the Lagragian energy functional (Π). Winkler and Pasternak foundations terms $(Kw - G\nabla^2 w)$ are attached in the radial direction and equations of motion of thin cylindrical shell are obtained in differential operator form as:

$$L_{11}u + L_{12}v + L_{13}w = \rho_t \frac{\partial^2 u}{\partial t^2}$$

$$L_{21}u + L_{22}v + L_{23}w = \rho_t \frac{\partial^2 v}{\partial t^2}$$

$$L_{31}u + L_{32}v + L_{33}w = \rho_t \frac{\partial^2 w}{\partial t^2} + Kw - G\nabla^2 w \qquad (8)$$

where $L_{ij}(i, j = 1,2,3)$ state the differential operators with regard to *x* and φ and are given in Appendix I and *G* and *K* stand for Pasternak and Winkler foundation modulii. Differential operator ∇^2 is defined as:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \varphi^2} \qquad (9)$$

Shell dynamical equations attain the following form after applying differential operators:

$$e_{11}\frac{\partial^2 u}{\partial x^2}+e_{12}\frac{\partial^2 u}{\partial \varphi^2}+e_{13}\frac{\partial^2 v}{\partial x\partial \varphi}+e_{14}\frac{\partial w}{\partial x}+e_{15}\frac{\partial^3 w}{\partial x^3}+e_{16}\frac{\partial^3 w}{\partial x\partial \varphi^2}=\rho_t\frac{\partial^2 u}{\partial t^2}$$

$$e_{21}\frac{\partial^2 u}{\partial x\partial \varphi}+e_{22}\frac{\partial^2 v}{\partial x^2}+e_{23}\frac{\partial^2 v}{\partial \varphi^2}+e_{24}\frac{\partial^3 w}{\partial x^2\partial \varphi}+e_{25}\frac{\partial w}{\partial \varphi}+e_{26}\frac{\partial^3 w}{\partial \varphi^3}=\rho_t\frac{\partial^2 v}{\partial t^2}$$

$$e_{31}\frac{\partial^3 u}{\partial x^3}+e_{32}\frac{\partial u}{\partial x}+e_{33}\frac{\partial^3 u}{\partial x\partial \varphi^2}+e_{34}\frac{\partial^3 v}{\partial x^2\partial \varphi}+e_{35}\frac{\partial^3 v}{\partial \varphi^3}+e_{36}\frac{\partial v}{\partial \varphi}+e_{37}\frac{\partial^4 w}{\partial x^4}+e_{38}\frac{\partial^4 w}{\partial x^2\partial \varphi^2}+e_{39}\frac{\partial^4 w}{\partial \varphi^4}+e_{310}\frac{\partial^2 w}{\partial x^2}+e_{311}\frac{\partial^2 w}{\partial \varphi^2}+e_{312}w=\rho_t\frac{\partial^2 w}{\partial t^2}+Kw-G\left(\frac{\partial^2 w}{\partial x^2}+e_{313}\frac{\partial^2 w}{\partial \varphi^2}\right) \quad (10)$$

where $e'_{ij}s$ are defined in Appendix II.

Numerical procedure: For the present shell problem, the wave propagation approach is used to investigate the vibration characteristics of three-layered cylindrical shells with isotropic material at the middle layer resting on Pasternak and Winkler elastic foundations. With the applications of present approach, the vibrations of functionally graded cylindrical shells are examined for simply supported-simply supported, clamped-clamped, clamped-simply supported and clamped-free boundary conditions. This approach is very convenient and easily can be applied to extract the shell vibration characteristics, saves time and a huge amount of algebraic expressions. To separate the time and space variables, the following modal displacement form is employed:

$$u(x,\varphi,t)=A\cos(n\varphi)e^{i(\omega t-k_m x)}$$
$$v(x,\varphi,t)=B\sin(n\varphi)e^{i(\omega t-k_m x)}$$
$$w(x,\varphi,t)=C\cos(n\varphi)e^{i(\omega t-k_m x)} \quad (11)$$

along the longitudinal, circumferential and transverse directions respectively. The constants A, B and C are the amplitudes of vibrations in the *x*, φ and *z* directions respectively, *n* is the number of circumferential waves and k_m stands for axial wave number associated with a boundary condition as given by (Zhang et al. 2001a). ω denotes the natural angular frequency of the cylindrical shell.

Derivation of frequency equation: Substitution of the shell deformation displacement functions *u*, *v* and *w* from the Eq. (11) in Eq. (10), the system of shell dynamic equations transformed into the following form:

$$c_{11}A+c_{12}B+c_{13}C=\omega^2\rho_t A \quad (12a)$$
$$c_{21}A+c_{22}B+c_{23}C=\omega^2\rho_t B \quad (12b)$$
$$c_{31}A+c_{32}B+c_{33}C+Kw+G(k_m^2+e_{313}n^2)=\omega^2\rho_t C \quad (12c)$$

where $c'_{ij}s(i,j=1,2,3)$ are described in Appendix III. The simultaneous algebraic equations (12a)-(12c) are written in matrix form as:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ -c_{21} & c_{22} & c_{23} \\ -c_{13} & c_{23} & c_{33} \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix}=\omega^2\rho_t\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad (13)$$

For non-trivial solution, the determinant of the matrix coefficients is vanished. This eigen - value problem is solved by using MATLAB software to extract the natural frequencies of the cylindrical shells.

RESULTS AND DISCUSSION

Simply supported cylindrical shells: In Table 1, the frequency parameters Ω for a cylindrical shell with simply supported boundary conditions, are compared with those results determined by (Zhang *et al*. 2001b). The geometrical parameters are assumed to be as: $L/R=20$, *h/R=0.05.* For the parameters L/R, a value of $m=1$ is used and n are selected from 0-4 in the comparison. The material properties of the shell are given as: $\rho=7850$ kg/m^3, $\nu=0.3$ and $E=2.1\times 10^{11}$ N/m^2.

Table 1: Comparison of frequency parameter $\Omega=\omega R\sqrt{(1-\nu^2)\rho/E}$ for a SS-SS cylindrical shell ($m=1,\ L/R=20,\ \nu=0.3$)

h/R	*n*	(Zhang *et al.*2001)	Present	% Difference
0.05	0	0.0929586	0.0929489	-0.0104348
	1	0.0161065	0.0161025	-0.0248347
	2	0.0393038	0.0392985	-0.0134847
	3	0.1098527	0.1098247	-0.0254887
	4	0.2103446	0.2102848	-

0.0000598

Clamped - clamped cylindrical shells:

In Table 2, there is a comparison of analytical frequency parameter (Ω). The numerical results for a clamped-clamped cylindrical shells evaluated by the present method are compared with those in (Naeem *et al.* 2009). The parameters used in this comparison are: $m = 1$, $L = 304.8$ mm, $h = 0.254$ mm, $R = 76.2$ mm, $E = 2.06 \times 10^{11}$N/m^2, $\upsilon = 0.3$, $\rho = 7.85 \times 10^3$ Kg/m^3 and *n* from 3 to 14 .

Table 2: Comparison of analytical frequency parameter $\Omega = \omega R\sqrt{(1-\nu^2)\rho / E}$ for a clamped-clamped (C-C) cylindrical shells ($m = 1$, $L = 304.8$ mm, $h = 0.254$ mm, $R = 76.2$ mm, $E = 2.06 \times 10^{11}$N/m^2, $\upsilon = 0.3$, $\rho = 7.85 \times 10^3$ Kg/m^3)

n	**(Naeem *et al.*2009)**	**Present**	**% Difference**
3	0.1033	0.1205	16.650
4	0.0684	0.0752	9.941
5	0.0517	0.0546	5.609
6	0.0476	0.0491	3.151
7	0.0529	0.0537	1.512
8	0.0640	0.0646	0.937
9	0.0788	0.0794	0.761
10	0.0964	0.0970	0.622
11	0.1162	0.1168	0.516
12	0.1381	0.1387	0.434
13	0.1620	0.1627	0.432
14	0.1880	0.1886	0.319

Clamped-simply supported cylindrical shells: Table 3 displays a comparison of frequency parameters $\Omega = \omega R\sqrt{(1-\nu^2)\rho / E}$ determined by the present method with ones determined by (Naeem *et al.* 2009) for a clamped-simply supported cylindrical shell. A good agreement is obvious between the two sets of results.

Table 3: Comparison of frequency parameter $\Omega = \omega R\sqrt{(1-\nu^2)\rho / E}$ for a clamped - simply supported cylindrical shells ($m = 1, L/R = 20, h/R = 0.002, \nu = 0.3$)

n	**(Naeem *et al.* 2009)**	**Present**	**%Difference**
1	0.024029	0.024721	2.88
2	0.008283	0.008282	-0.01
3	0.005844	0.005852	0.14
4	0.008705	0.008710	0.05
5	0.013678	0.013684	0.04
6	0.019973	0.019979	0.03
7	0.027459	0.027466	0.02
8	0.036111	0.036118	0.02
9	0.045984	0.045929	-0.12

10	0.056889	0.056897	0.01

Clamped - free cylindrical shells: In Table 4, frequency parameters (Ω) for a clamped-free cylindrical shell are compared with those given by (Naeem and Sharma, 1999). Naeem and Sharma employed the Ritz formulation to investigate the shell problem, whereas the wave propagation technique is used to calculate shell frequencies. A good agreement between the results is noticed. There is once again an excellent agreement between the two sets of analytical results.

Table 4: Convergence of frequency parameter $\Omega = \omega R\sqrt{(1-\nu^2)\rho / E}$ for clamped-free (C-F) cylindrical shell ($m = 1$, $\frac{L}{R} = 6$, $\frac{h}{R} = 0.002$, $\nu = 0.3$)

N	**(Naeem and Sharma, 1999)**	**Present**	**% Difference**
1	0.055904	0.042417	-24.13
2	0.020086	0.014424	-28.19
3	0.010636	0.008137	-23.49
4	0.010137	0.009314	-8.119
5	0.014108	0.013862	-1.744
6	0.020138	0.020050	-0.437
7	0.027541	0.027504	-0.022
8	0.036162	0.036145	-0.047
9	0.045958	0.045951	-0.015
10	0.056917	0.056916	-0.002

From the above comparisons of numerical results, it is noticed that the method employed here is very efficient, valid, fast and provides accurate results.

Three - layered cylindrical shells: A number of results for the proposed three-layered cylindrical shell with middle layer consist of functionally graded material, as shown in Fig. 1, are determined for various sets of material and geometrical parameters. The inner and outer layers of the shell are comprised of isotropic material whereas the middle layer is assumed to be functionally graded Material properties of shell are represented by Young's modulus (E), Poisson's ratio (ν) and mass density (ρ).

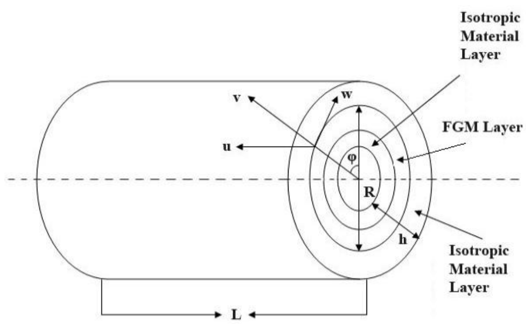

Fig.1 Geometry of three layered cylindrical shell

In general, vibration characteristics are most influenced by Young's modulus. In this study, the Poisson's ratio is presumed to be constant for functionally graded materials whereas the Young's modulus dependents on intrinsic thickness variable (z) as well as the Young's modulus of constituent materials forming functionally graded layers. Here two configurations of a cylindrical shell are considered to suggest with regard to the shell layer thickness. In first configuration, the thickness of each layer is supposed to be of *h/3* while in the second configuration, the thickness of each of the inner and outer layers are of *h/4* and that of middle layer is of *h/2*.

The stiffness moduli A_{ij} , B_{ij} and D_{ij} are modified in according to the thickness of material layers when inner and outer layers are isotropic and middle is functionally graded as:

$$A_{ij} = A_{ij}^{in(isotropic)} + A_{ij}^{m(FG)} + A_{ij}^{out(isotropic)}$$
$$B_{ij} = B_{ij}^{in(Isotropic)} + B_{ij}^{m(FG)} + B_{ij}^{out(isotropic)}$$
$$D_{ij} = D_{ij}^{in(isotropic)} + D_{ij}^{m(FG)} + D_{ij}^{out(isotropic)}$$
(14)

where $i, j = 1,2,6$ and in(isotropic), m(FG), out(isotropic), are associated with inner isotropic, middle functionally graded, outer isotropic layers of cylindrical shell respectively. Here by considering the constituent materials of stainless steel and aluminum for isotropic layers and also the FG layers are structured from two kinds of materials, nickel and zirconia. In this way, four types of shells are obtained and are listed in the following Table 5.

Table 5: Description of cylindrical shells

Types	**Isotropic Inner**	**FGM Material Middle**	**Isotropic outer**
Type 1	Stainless Steel	Zirconia - Nickel	Stainless Steel
Type 2	Stainless Steel	Nickel - Zirconia	Stainless Steel
Type3	Stainless Steel	Zirconia - Nickel	Aluminum
Type 4	Stainless Steel	Nickel - Zirconia	Aluminum

Material properties of isotropic materials: Steel and Aluminum are given in Table 6 whereas the material properties of the constituent materials forming functionally graded layers are listed in Table 7.

Table 6: Material properties of isotropic materials

Isotropic	$E(N/m^2)$	Poisson ratio(ν)	Density ρ (Kg/m^3)
Stainless Steel	68.95E+09	0.315	2.7145E+03
Aluminum	2.1E+11	0.28	7.8E+03

Table 7: Material properties of nickel and zirconia

FGM	$E(N/m^2)$	Poisson ratio(ν)	Density ρ (Kg/m^3)
Nickel	2.05098E+11	0.3100	8900
Zirconia	1.6806296E+11	0.297996	5700

Frequency analysis of cylindrical shells: In this section, variation of frequencies for four types of cylindrical shells described above are analyzed for three end conditions viz, clamped - clamped (C - C), clamped - simply supported (C - SS) and clamped - free (C - F).

Variations of natural frequencies (Hz) against circumferential wave number (n): In Tables 8-10 , natural frequencies (Hz) of four types of cylindrical shells are given against the circumferential wave numbers (*n*). Clamped - clamped , clamped - simply supported and clamped - free boundary conditions applied on the shell ends. The shell parameter data is $m = 1, L/R = 20, h/R = 0.002$. For functionally graded layer, the power law exponent *p*, is 5. The numerical results exhibit the well-known characteristic of shell vibration frequencies, i.e., the value of the frequency first decreases achieves its lowest value and then it begins to increase with the circumferential wave modes (*n*). For Type 1 and 2, the outer and inner layers are of steel. This shows that the values of shell frequencies increase with interchanging the order of constituent materials forming the functionally graded middle layer. For Type 1, the inner and outer layer is of steel whereas in Type 3, inner is of steel and outer is of aluminum. The frequency increases with the interchange of inner and outer layers material whereas the configuration of functionally graded layer is same. Also in Type 4, the inner and outer layers posses the same materials as in Type 3 whereas the materials of middle layer has been replaced by changing the order of zirconia and nickel. This shows that the frequency has been increased by changing the materials of functionally graded layer. The same behaviour of shell frequency variations with circumferential waves (*n*) is observed by these four types of cylindrical shells.

Table 8: Variation of natural frequencies (Hz) of a three-layered clamped-clamped cylindrical shell (*m=1,p=5,L/R=20,h/R=0.002*)

n	Type 1	Type 2	Type 3	Type 4
1	28.7544	30.0925	29.2850	30.2737
2	9.6851	10.1040	9.8754	10.1878
3	5.4815	5.7173	5.8350	6.0336
4	6.2720	6.6812	7.1909	7.5488
5	9.3122	10.0285	10.8915	11.5040

Table 9: Variation of natural frequencies (Hz) of a three-layered clamped-simply supported cylindrical shell (*m=1,p=5,L/R=20,h/R=0.002*)

n	Type 1	Type 2	Type 3	Type 4
1	20.3824	21.3296	20.7566	21.4563
2	6.8050	7.0906	6.9508	7.1653
3	4.3715	4.5802	4.7682	4.9502
4	5.9620	6.3855	6.9135	7.2803
5	9.2175	9.9470	10.8078	11.4283

Table 10: Variation of natural frequencies (Hz) of a three-layered clamped-free cylindrical shell (*m=1,p=5,L/R=20,h/R=0.002*)

n	Type 1	Type 2	Type 3	Type 4
1	3.4048	3.5566	3.4657	3.5781
2	1.5151	1.5837	1.6558	1.7167
3	2.9916	3.2221	3.5009	3.6980
4	5.6472	6.1099	6.6364	7.0270
5	9.1169	9.8738	10.7193	11.3562

In Table 11, variations of natural frequencies for Type 1 for two configurations with respect to thickness of layers, are shown against circumferential wave numbers (n).

Table 11: Variation of natural frequencies (Hz) of a three-layered clamped-clamped cylindrical shell for two configurations with respect to thickness of shell layers (*m=1,p=5,L/R=20,h/R=0.002*)

n	Type 1 (*h/3*)	Type 1 (*h/4*)	% Difference
1	28.7544	28.6163	0.4803
2	9.6851	9.6514	0.3480
3	5.4815	5.4589	0.4123
4	6.2720	6.1775	1.5067
5	9.3122	9.1194	2.0704

It is seen from the Table 11, by decreasing the thickness of the middle functionally graded layer, the frequencies diminish.

Frequency analysis of cylindrical shells with elastic foundations: Fig.2 (a)-(c) represent variations of natural frequencies (Hz) for three layered cylindrical shells for C - C, C - SS and C - F edge conditions respectively with elastic foundations, i.e., $G = 1 \times 10^9$, $K = 1 \times 10^9$ N-m. The middle layer is functionally graded material. For both Types 1 and 2 of the cylindrical shells, the frequencies increase considerably by adding the elastic foundations, i.e., Pasternak and Winkler models. Also, the values of natural frequencies increase gradually against circumferential wave modes (n). The minimum frequency is associated with *n=1*. This shows that the shell vibration is similar to that of beam type for these boundary conditions.

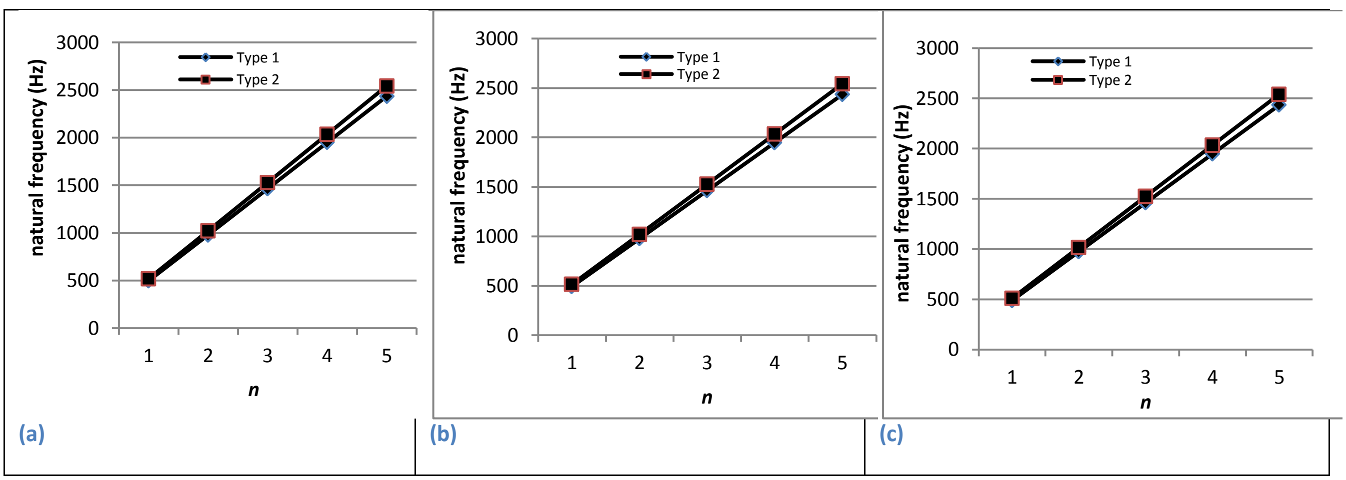

Fig. 2 Variations of natural frequencies (Hz) for three layered (a) C-C (b) C-SS (c) C-F cylindrical shells, ($G = 1 \times 10^9$ N-m, $K = 1 \times 10^9$ N-m, , $\frac{L}{R} = 20, \frac{h}{R} = 0.002, p = 10$)

In Table 12, natural frequencies (Hz) for C - C, C - SS and C - F three layered cylindrical shells are listed for Type 1,2,3,4 cylindrical shells. The shells are based on the Pasternak and Winkler foundations. The values of the shell frequencies show considerable increments by inducting the elastic moduli. The Type 2 has the highest frequency, followed by Type 1, Type 4, and Type 3. This also exhibits the effects of material thickness of the layers forming cylindrical shell. The clamped-free cylindrical shells have the lowest frequency.

Table 12: Variation of natural frequencies (Hz) of a clamped-clamped (C-C), clamped-simply supported (C-SS) and clamped-free (C-F) three-layered cylindrical shells on elastic foundations ($m = 1$, $p = 1$, $L = 0.41$m, $R = 0.3015$m, $h = 0.001$m, $\nu = 0.3$, $G = 1.5 \times 10^7$N-m, $K = 2.5 \times 10^7$N-m)

Boundary Condition	n	Type 1	Type 2	Type 3	Type 4
	1	11913.3	11920.6	10623.8	10640.6
	2	12308.9	12324.1	11616.7	11637.0
C-C	3	13399.2	13427.5	13210.3	13233.1
	4	15186.8	15222.1	15180.8	15206.5
	5	17318.5	17359.2	17389.3	17418.3
	1	10124.1	10132.6	9114.88	9129.84
C-SS	2	10720.9	10738.8	10277.8	10295.9
	3	12158.3	12185.9	12060.4	12081.1
	4	14170.7	14203.9	14193.9	14217.8
	5	16452.6	16491.3	16534.6	16562.1
	1	5387.34	5398.30	5156.02	5165.03
C-F	2	7150.12	7166.74	7109.50	7121.60
	3	9486.56	9508.74	9518.33	9534.16
	4	12031.8	12059.9	12103.6	12123.6
	5	14671.4	14705.7	14773.8	14798.2

Figs. 3(a)-(d), demonstrate the values of natural frequencies (Hz) for Types 1,2,3,4 of cylindrical shells against the boundary conditions considered to be SS – SS, C – SS, C – C and C – F. the elastic foundations are $G = 0$ and $K = 2.5e + 07$ N-m. Circumferential wave mode *n* and axial wave mode *m* are taken both 1. Radius-to-thickness *h/R* ratio is assumed to be 0.004. In each figure, four frequency curves related to the boundary conditions are separated at the start but converge as *L/R* is increased and mingle into a single curve at $L/R > 12$.

Figs. 4(a)-(d), exhibit variations of natural frequencies of cylindrical shells: Type1, Type 2, Type 3 and Type 4, against thickness-to-radius ratios (*h/R*) assuming circumferential wave mode *n* and axial wave mode $m = 1$, based on Pasternak $G = 0$ and Winkler $K = 1.5e + 07$ N-m foundations. The length-to-thickness ratio is taken as 10. The four cylindrical shells are supported by the boundary conditions SS - SS, C - SS, C - C and C - F respectively. The four frequency curves are separated for each type of cylindrical shell. The frequencies for the clamped-clamped cylindrical shell are the largest, then followed by the clamped - simply supported, simply supported - simply supported and clamped – free cylindrical shells. This behaviour is due to change in geometric constraints in different edge conditions. The above discussion shows that the influence of Pasternak and Winkler elastic moduli on the shell frequency is pronounced along with the other shell parameters.

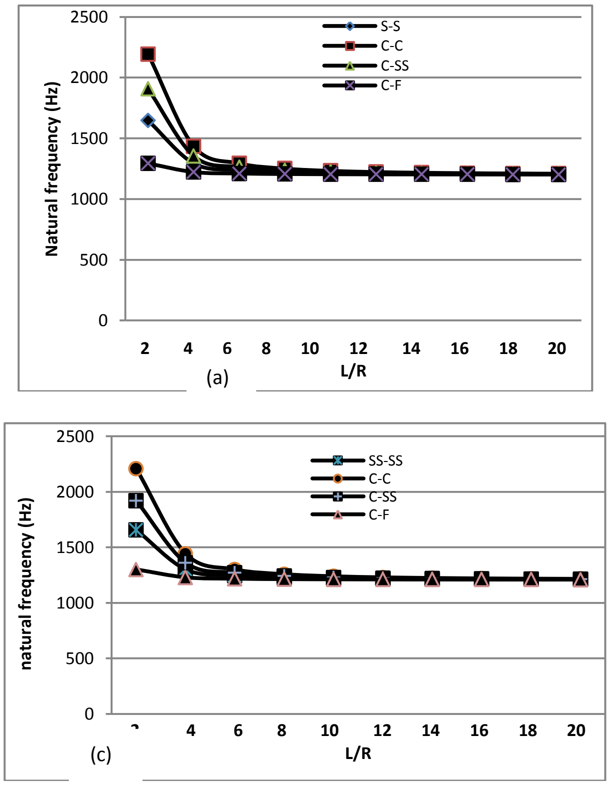

Fig. 3 Variations of natural frequencies (Hz) of (a) Type1 (b) Type 2 (c Type 3 (d) Type 4 cylindrical shells for $G = 0,\ K = 2.5 \times 10^7$ **N-m,** $h = 0.004, m = 1, R = 1, n = 1$

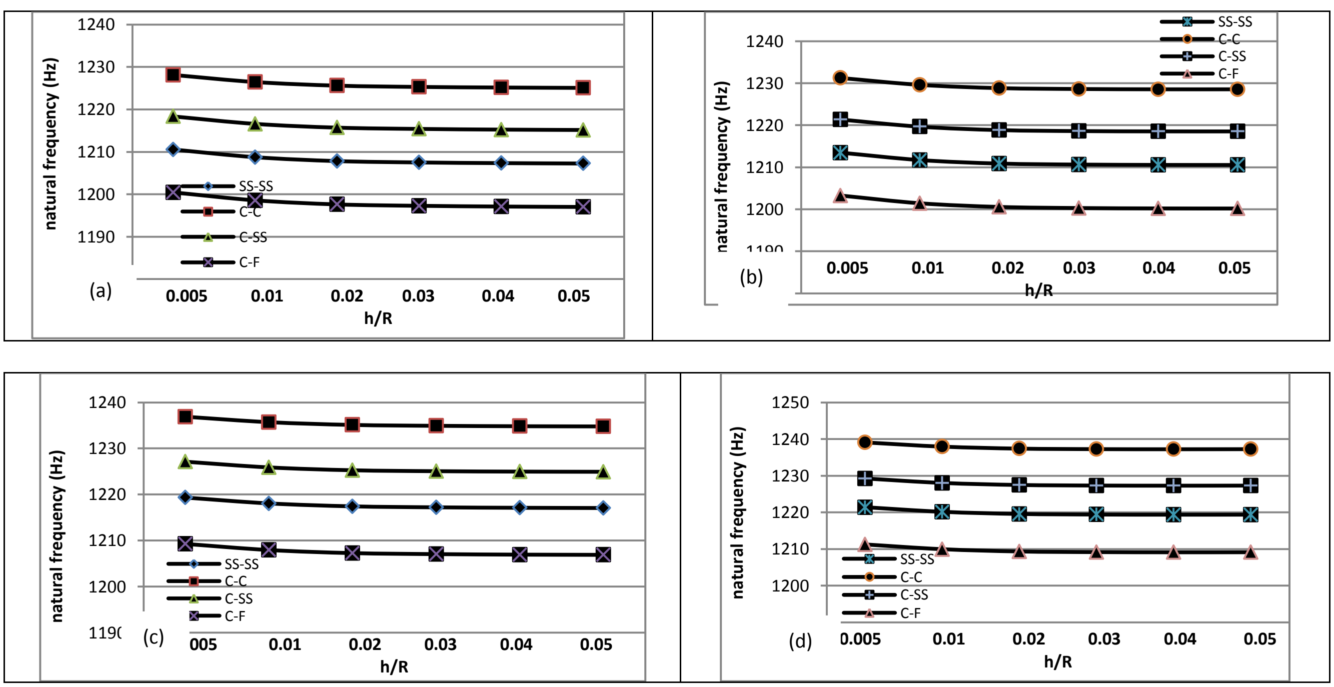

Fig. 4 Variations of natural frequencies (Hz) of (a) Type1 (b) Type 2 (c Type 3 (d) Type 4 cylindrical shells for $\boldsymbol{G = 0, K = 1.5 \times 10^7}$ **N-m,** $\boldsymbol{L = 10, m = 1, R = 1, n = 1}$

Conclusion: In this study, vibrations of three-layered cylindrical shells are investigated under various boundary conditions with middle layer is assumed to be of functionally graded. Pasternak and Winkler foundations are appended in the transverse direction. Wave propagation approach is utilized to frame the shell frequency equation. The influences of the configurations of the cylindrical shells are analyzed with interchange of materials of shell layers. It is observed that the value of the frequency first decreases, achieves its lowest value and then it begins to increase with the circumferential wave modes. It is seen that frequencies are influenced when materials of inner and outer isotropic layers or composition of middle FGM layer of cylindrical shell are interchanged. It is further seen that by decreasing the thickness of the middle functionally graded layer, the frequencies diminish. Moreover, frequencies increase considerably by adding the elastic foundations, and the influence of Pasternak is more pronounced than Winkler on the shell frequency.

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Appendix I

$$L_{11} = A_{11}\frac{\partial^2}{\partial x^2} + \frac{A_{66}}{R^2}\frac{\partial^2}{\partial \varphi^2},$$
$$L_{12} = \frac{(A_{12}+A_{66})}{R}\frac{\partial^2}{\partial x \partial \varphi} + \frac{(B_{12}+2B_{66})}{R^2}\frac{\partial^2}{\partial x \partial \varphi},$$
$$L_{13} = \frac{A_{12}}{R}\frac{\partial}{\partial x} - B_{11}\frac{\partial^3}{\partial x^3} - \frac{(B_{12}+2B_{66})}{R^2}\frac{\partial^3}{\partial x \partial \varphi^2},$$
$$L_{21} = \frac{(A_{12}+A_{66})}{R}\frac{\partial^2}{\partial x \partial \varphi} + \frac{(B_{12}+B_{66})}{R^2}\frac{\partial^2}{\partial x \partial \varphi},$$
$$L_{22} = \left(A_{66} + \frac{3B_{66}}{R} + \frac{2D_{66}}{R^2}\right)\frac{\partial^2}{\partial x^2} + \left(\frac{A_{22}}{R^2} + \frac{2B_{22}}{R^3} + \frac{D_{22}}{R^4}\right)\frac{\partial^2}{\partial \varphi^2},$$
$$L_{23} = \left(\frac{A_{22}}{R^2} + \frac{B_{22}}{R^3}\right)\frac{\partial}{\partial \varphi} - \left(\frac{B_{22}}{R^3} + \frac{D_{22}}{R^4}\right)\frac{\partial^3}{\partial \varphi^3} - \left(\frac{B_{12}+2B_{66}}{R} + \frac{D_{12}+2D_{66}}{R^2}\right)\frac{\partial^3}{\partial x^2 \partial \varphi},$$
$$L_{31} = -\frac{A_{12}}{R}\frac{\partial}{\partial x} + B_{11}\frac{\partial^3}{\partial x^3} + \frac{(B_{12}+2B_{66})}{R^2}\frac{\partial^3}{\partial x \partial \varphi^2},$$
$$L_{32} = -\left(\frac{A_{22}}{R^2} + \frac{B_{22}}{R^3}\right)\frac{\partial}{\partial \varphi} + \left(\frac{B_{22}}{R^3} + \frac{D_{22}}{R^4}\right)\frac{\partial^3}{\partial \varphi^3} + \left(\frac{B_{12}+2B_{66}}{R} + \frac{D_{12}+4D_{66}}{R^2}\right)\frac{\partial^3}{\partial x^2 \partial \varphi},$$
$$L_{33} = -\frac{A_{22}}{R^2} + \frac{2B_{12}}{R}\frac{\partial^2}{\partial x^2} + \frac{2B_{22}}{R^3}\frac{\partial^2}{\partial \varphi^2} - D_{11}\frac{\partial^4}{\partial x^4} - 2\frac{D_{12}+2D_{66}}{R^2}\frac{\partial^4}{\partial x^2 \partial \varphi^2} - \frac{D_{22}}{R^4}\frac{\partial^4}{\partial \varphi^4},$$

Appendix II

$$e_{11} = A_{11}, e_{12} = \frac{A_{66}}{R^2}, e_{13} = \frac{A_{12}+A_{66}}{R} + \frac{2B_{66}+B_{12}}{R^2}, e_{14} = \frac{A_{12}}{R}, e_{15} = -B_{11},$$
$$e_{16} = \frac{-B_{12}+2B_{66}}{R^2}, e_{21} = \frac{A_{12}+A_{66}}{R} + \frac{B_{66}+B_{12}}{R^2}, e_{22} = A_{66} + \frac{3B_{66}}{R} + \frac{2D_{66}}{R^2},$$
$$e_{23} = \frac{A_{22}}{R^2} + \frac{2B_{22}}{R^3} + \frac{D_{22}}{R^4}, e_{24} = -\left(\frac{B_{12}+2B_{66}}{R} + \frac{2D_{66}+D_{12}}{R^2}\right), e_{25} = \frac{A_{22}}{R^2} + \frac{B_{22}}{R^3},$$
$$e_{26} = -\left(\frac{B_{22}}{R^3} + \frac{D_{22}}{R^4}\right), e_{31} = B_{11} = -e_{15}, e_{32} = -\frac{A_{12}}{R} = -e_{14}, e_{33} = \frac{B_{12}+2B_{66}}{R^2},$$
$$e_{34} = \left(\frac{B_{12}+2B_{66}}{R} + \frac{4D_{66}+D_{12}}{R^2}\right) = -e_{24} + \frac{2D_{66}}{R^2}, e_{35} = \left(\frac{B_{22}}{R^3} + \frac{D_{22}}{R^4}\right) = -e_{26},$$
$$e_{36} = -\left(\frac{A_{22}}{R^2} + \frac{B_{22}}{R^3}\right) = -e_{25}, e_{37} = -D_{11}, e_{38} = -\left(\frac{2D_{12}+4D_{66}}{R^2}\right), e_{39} = -\frac{D_{22}}{R^4},$$
$$e_{310} = \frac{2B_{12}}{R}, e_{311} = \frac{2B_{22}}{R^3}, e_{312} = -\frac{A_{22}}{R^2}, e_{313} = \frac{1}{R^2}$$

Appendix III

$$c_{11} = k_m^2 e_{11} + n^2 e_{12},$$
$$c_{12} = ink_m e_{13},$$
$$c_{13} = ik_m(e_{14} - n^2 e_{16} - k_m^2 e_{15}),$$
$$c_{21} = -ink_m e_{21},$$
$$c_{22} = k_m^2 e_{22} + n^2 e_{23},$$
$$c_{23} = -nk_m^2 e_{24} + ne_{25} - n^3 e_{26},$$
$$c_{31} = -ik_m(e_{14} - n^2 e_{16} - k_m^2 e_{15}),$$
$$c_{32} == -nk_m^2 e_{24} + ne_{25} - n^3 e_{26},$$
$$c_{33} = e_{312} + n^2 e_{311} + k_m^2 e_{310} - n^4 e_{39} - nk_m^2 e_{38} - k_m^4 e_{37} + K + G(k_m^2 + e_{313} n^2).$$