FINITE ELEMENT MODEL FOR LINEAR SECOND ORDER ONE DIMENSIONAL HOMOGENEOUS TELEGRAPH EQUATION

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ABSTRACT: The Telegraph equation arises in the propagation of electrical signals along a transmission line and wave phenomena. Interaction between convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear physical, chemical and biological processes. Linear second order hyperbolic partial differential equations describe various phenomena in acoustics, electromagnetic and fluid dynamics. In this paper, a Galerkin based Finite Element Model has been developed to solve linear second order one dimensional Telegraph equation numerically. Accuracy of the developed scheme has been analyzed by comparing the numerical solution with exact solution.

Keywords: Finite Element Model, Galerkin Method, Lagrangian polynomials, Shape functions.

INTRODUCTION

Partial Differential Equations (PDE,s) are at the heart of many, if not most, computer analyses or simulations of continuous physical systems, such as fluids, electromagnetic fields, the human body and so on [William]. A class of hyperbolic Partial Differential Equations which describes vibrations with in objects and how waves are propagated is called Telegraph equation [Gerald]. In physics, propagation of sound, light and water waves is modeled by hyperbolic partial differential equations. The efficient and accurate numerical techniques for the Telegraph equations are of fundamental importance for the simulation of time dependent of transmission line and wave phenomena. Finite difference methods are commonly used for the simulations of time dependent waves because of their simplicity and their efficiency on structured Cartesian meshes [Kries, Burden, and Hoffman]. However in presence of complex geometry, their usefulness is somewhat limited. In contrast Finite Element Methods [Reddy, Sastry] can easily handle these cases. Moreover their extension to higher order is straightforward. In this paper, a Finite Element Model for linear second order one dimensional Telegraph equation has been developed. Galerkin method has been used to setup the element equations and a central finite difference scheme has been used to approximate the first and second order time derivatives.

Finite Element Model: Let us consider the second order one dimensional Telegraph equation

$$\alpha\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial u}{\partial t}+\gamma u=c^2\frac{\partial^2 u}{\partial x^2}\quad 0<x<l,\ t>0 \tag{1}$$

with boundary conditions

$$u(0,t)=u(l,t)=0,\qquad t>0$$

and initial conditions

$$u(x,0)=f(x),\qquad 0\le x\le l$$

and

$$\frac{\partial u}{\partial t}(x,0)=g(x),\qquad 0\le x\le l$$

Domain Discretization: Let us consider the global domain as shown in Figure 1, in which we have to approximate the solution of equation (1). We divide the global domain into finite number of rectangular elements. Let there be *K* nodes and *K*-1 linear elements in spatial direction. In the Figure 1, i represents i^{th} node and (i) represents the i^{th} element. Each element has two nodes, e.g. element (i) has left node i and right node $i+1$. The length of element (i) is given by $\Delta x_i = x_{i+1}-x_i$. In a similar way we take an element along temporal axis each of length $\Delta t^n = t^{n+1}-t^n$.

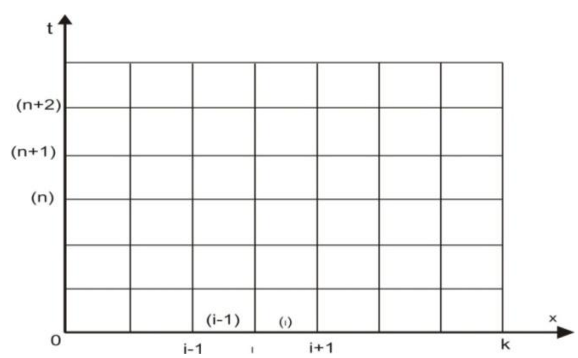

Figure 1

Interpolating Functions: Let us approximate the solution of equation (1) by $u\ (x,t)$, where $u\ (x,t)$ is defined as

$$u(x,t) = u^{(1)}(x,t) + \dots + u^{(i)}(x,t) + \dots + u^{(K-1)}(x,t) \quad (2)$$

where each $u^{(i)}(x,t) \quad (i = 1,2,3,\dots.K-1)$ represents the local interpolating polynomials over the element (i). Write $u^{(i)}(x,t)$ for i^{th} element as

$$u^{(i)}(x,t) = u_i(t)N_i^{(i)}(x) + u_{i+1}(t)N_{i+1}^{(i)}(x) \quad (3)$$

Where $u_i(t) = (i = 1,2,\dots K)$ represents the nodal values and $N_i^{(i)}$ and $N_i^{(i+1)}$ represent the shape functions for the element (i) at nodes i and $i+1$ respectively and $N_i{}'s$ are Lagrangian Polynomials of degree one. We define

$$N_i^{(i)}(x) = -\frac{x-x_{i+1}}{x_{i+1}-x_i} = -\frac{x-x_{i+1}}{\Delta x_i} \quad (4)$$

and

$$N_{i+1}^{(i)}(x) = \frac{x-x_i}{x_{i+1}-x_i} = \frac{x-x_i}{\Delta x_i} \quad (5)$$

Substituting the values from (4) and (5) into equation (3), we have

$$u^{(i)}(x,t) = u_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + u_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right) \quad (6)$$

Element Equations: In this section we will use the Galerkin method to approximate the solution of one dimensional Telegraph equation given by (1) i.e

$$\alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} + \gamma u = c^2\frac{\partial^2 u}{\partial x^2}$$

Whose residual is

$$R(x,t) = \alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} + \gamma u - c^2\frac{\partial^2 u}{\partial x^2} \quad (7)$$

Let us define the integral $I\,(u(x,t))$ of weighted residual, which is developed by multiplying $R(x,t)$ by weighting functions $W_k\,(x)\;(k = 1,2,3,\dots..)$ and integrating that integral over the entire domain. Then set this integral equal to zero. We take the general weighting function $W(x)$. Therefore

$$I\,(u(x,t)) = \int_a^b W\left(\alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} + \gamma u - c^2\frac{\partial^2 u}{\partial x^2}\right)dx = 0$$

$$I\,(u(x,t)) = \int_a^b \alpha W\frac{\partial^2 u}{\partial t^2}dx + \int_a^b \beta W\frac{\partial u}{\partial t}dx - \int_a^b c^2 W\frac{\partial^2 u}{\partial x^2}dx + \int_a^b \gamma W u\,dx = 0 \quad (8)$$

Now solving the third integral in (8) by parts

That is

$$\int_a^b c^2 W\frac{\partial^2 u}{\partial x^2}dx = \left[c^2 W\frac{\partial u}{\partial x}\right]_a^b - \int_a^b c^2\frac{\partial W}{\partial x}\frac{\partial u}{\partial x}dx \quad (9)$$

The factor $\frac{\partial u}{\partial x}$ in first term on R.H.S of (9) cancels out at all interior points when we assemble the element equations. It exists only at first and last node when there are boundary conditions on derivatives. Therefore we will drop this term so that equation (8) will take the form

$$I\,(u(x,t)) = \int_a^b \alpha W\frac{\partial^2 u}{\partial t^2}dx + \int_a^b \beta W\frac{\partial u}{\partial t}dx + \int_a^b c^2\frac{\partial W}{\partial x}\frac{\partial u}{\partial x}dx + \int_a^b \gamma W u\,dx = 0 \quad (10)$$

Now the weighted residual integral $I\,(u(x,t))$ for the entire domain is expressed as sum of weighted residual integrals of each element $(i)\;(i = 1,2,\dots K)$

$$I\,(u(x,t)) = I^{(1)}(u(x,t)) + \dots + I^{(i)}(u(x,t)) + \dots + I^{(K-1)}(u(x,t)) = 0 \quad (11)$$

where

$$I^{(i)}(u(x,t)) = \int_{x_i}^{x_{i+1}} \alpha W\frac{\partial^2 u}{\partial t^2}dx + \int_{x_i}^{x_{i+1}} \beta W\frac{\partial u}{\partial t}dx + \int_{x_i}^{x_{i+1}} c^2\frac{\partial W}{\partial x}\frac{\partial u}{\partial x}dx + \int_{x_i}^{x_{i+1}} \gamma W u\,dx = 0 \quad (12)$$

To evaluate $I^{(i)}(u(x,t))$ given by equation (12), we require $u^{(i)}(x,t)$ and its partial derivatives w.r.t $'t'$ and $'x'$. Differentiating (3) two times partially w.r.t 't' and substituting values of $N_i{}'s$ from (4) and (5) we get

$$\frac{\partial u^{(i)}}{\partial t} = \dot{u}_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + \dot{u}_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right) \quad (13)$$

$$\frac{\partial^2 u^{(i)}}{\partial t^2} = \ddot{u}_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + \ddot{u}_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right) \quad (14)$$

Now differentiating (3) w.r.t 'x'

$$\frac{\partial u^{(i)}}{\partial x} = u_i\left(-\frac{1}{\Delta x_i}\right) + u_{i+1}\left(\frac{1}{\Delta x_i}\right)$$

$$\frac{\partial u^{(i)}}{\partial x} = \frac{1}{\Delta x_i}(u_{i+1} - u_i) \quad (15)$$

Substituting the values from (13), (14) and (15) into equation (12)

$$I^{(i)}(u(x,t)) = \int_{x_i}^{x_{i+1}} \alpha W\left[\ddot{u}_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + \ddot{u}_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right)\right]dx + \int_{x_i}^{x_{i+1}} \beta W\left[\dot{u}_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + \dot{u}_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right)\right]dx + \int_{x_i}^{x_{i+1}} \gamma\frac{\partial W}{\partial x}\left[u_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + u_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right)\right]dx + \int_{x_i}^{x_{i+1}} c^2\frac{\partial W}{\partial x}\left[\frac{u_{i+1}-u_i}{\Delta x_i}\right]dx = 0 \quad (16)$$

Writing equation (16) symbolically as

$$I^{(i)}(u(x,t)) = A + B + C + D \quad (17)$$

where A, B, C and D represents the integrals in equation (17). In Galerkin method, the weighted functions $W_k\;(k = 1,2,\dots\dots)$ are considered to be shape functions. Here $N_i^{(i)}(x)$ and $N_{i+1}^{(i)}(x)$ are the shape function. Therefore let

$$W(x) = N_i^{(i)}(x) = -\frac{x-x_{i+1}}{\Delta x_i} \quad \text{and} \quad \frac{\partial W}{\partial x} = -\frac{1}{\Delta x_i}$$

Substituting the value of $W(x)$ and $\frac{\partial W}{\partial x}$ in equation (16) and solving the integrals for A, B, C and D

$$A = \int_{x_i}^{x_{i+1}} \alpha\left[-\frac{x-x_{i+1}}{\Delta x_i}\right]\left[\ddot{u}_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + \ddot{u}_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right)\right]dx$$

$$A = \alpha\frac{\Delta x_i}{6}[2\ddot{u}_i + \ddot{u}_{i+1}] \quad (18)$$

Next solving for B

$$B = \int_{x_i}^{x_{i+1}} \beta\left[-\frac{x-x_{i+1}}{\Delta x_i}\right]\left[\dot{u}_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + \dot{u}_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right)\right]dx$$

$$B = \beta\frac{\Delta x_i}{6}[2\dot{u}_i + \dot{u}_{i+1}] \quad (19)$$

Solving for the value of C

$$C = \int_{x_i}^{x_{i+1}} \gamma \left[-\frac{x-x_{i+1}}{\Delta x_i}\right]\left[u_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + u_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right)\right] dx$$

$$C = \frac{\gamma(\Delta x_i)}{6}[2u_i + u_{i+1}] \quad (20)$$

Now for D

$$D = \int_{x_i}^{x_{i+1}} c^2 \left[-\frac{1}{\Delta x_i}\right]\left[\frac{u_{i+1}-u_i}{\Delta x_i}\right] dx$$

$$D = -\frac{c^2}{\Delta x_i}[u_{i+1} - u_i] \quad (21)$$

Now substitute the values from $(18), (19), (20)$ and (21) in (17) we have

$$I^{(i)}(u(x,t)) = \alpha\frac{\Delta x_i}{6}[2\ddot{u}_i + \ddot{u}_{i+1}] + \beta\frac{\Delta x_i}{6}[2\dot{u}_i + \dot{u}_{i+1}] + \frac{\gamma(\Delta x_i)}{6}[2u_i + u_{i+1}] - \frac{c^2}{\Delta x_i}[u_{i+1} - u_i] = 0 \quad (22)$$

Further we let, $W(x) = N_{i+1}^{(i)}(x)$ then

$$W(x) = \frac{x-x_i}{\Delta x_i} \text{ and } \frac{\partial W}{\partial x} = \frac{1}{\Delta x_i}$$

Next we use these values of $W(x)$ and $\frac{\partial W}{\partial x}$ equation (17), and solving for A, B, C, D we get.

$$A = \int_{x_i}^{x_{i+1}} \left[\frac{x-x_i}{\Delta x_i}\right]\left[\ddot{u}_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + \ddot{u}_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right)\right] dx$$

$$A = \alpha\frac{\Delta x_i}{6}[\ddot{u}_i + 2\ddot{u}_{i+1}] \quad (23)$$

Now

$$B = \int_{x_i}^{x_{i+1}} \left[\frac{x-x_i}{\Delta x_i}\right]\left[\dot{u}_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + \dot{u}_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right)\right] dx$$

$$B = \beta\frac{\Delta x_i}{6}[\dot{u}_i + 2\dot{u}_{i+1}] \quad (24)$$

Solving for C we have

$$C = \int_{x_i}^{x_{i+1}} \gamma\left[\frac{x-x_i}{\Delta x_i}\right]\left[u_i\left(-\frac{x-x_{i+1}}{\Delta x_i}\right) + u_{i+1}\left(\frac{x-x_i}{\Delta x_i}\right)\right] dx$$

$$C = \gamma\frac{\Delta x_i}{6}[u_i + 2u_{i+1}] \quad (25)$$

$$D = \int_{x_i}^{x_{i+1}} c^2\left[\frac{1}{\Delta x_i}\right]\left[\frac{u_{i+1}-u_i}{\Delta x_i}\right] dx$$

$$D = c^2\left[\frac{u_{i+1}-u_i}{\Delta x_i}\right] \quad (26)$$

Now substituting values from equations $(23), (24), (25)$ and (26) into equation (17), we have:

$$I^{(i)}(u(x,t)) = \alpha\frac{\Delta x_i}{6}[\ddot{u}_i + 2\ddot{u}_{i+1}] + \beta\frac{\Delta x_i}{6}[\dot{u}_i + 2\dot{u}_{i+1}] + \frac{\gamma(\Delta x_i)}{6}[u_i + 2u_{i+1}] + \frac{c^2}{\Delta x_i}[u_{i+1} - u_i] = 0 \quad (27)$$

Equations (22) and (27) are the element equations for i^{th} element.

Assembly of Element Equations: Since i^{th} node is common between (i) and $(i-1)$ element therefore in order to get the nodal equation for i^{th} node we assemble the elements equation for node *i* in (i) and $(i-1)$ element. The physical domain for element (i) and $(i-1)$ is shown in Figure 2.

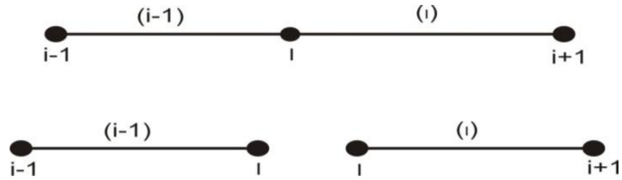

Figure2

As we can see from the figure 2 that the node $i+1$ for the element (i) corresponds to node i for the element $(i-1)$ and node i in element (i) corresponds to node $i-1$ in the element $(i-1)$. Therefore to write the element equations for node i of element $(i-1)$ we will replace i by $i-1$ in the element equation of element (i) for the node $i+1$. Thus we have from equation (27):

$$\frac{\alpha(\Delta x_{i-1})}{6}[\ddot{u}_{i-1} + 2\ddot{u}_i] + \frac{\beta(\Delta x_{i-1})}{6}[\dot{u}_{i-1} + 2\dot{u}_i] + \frac{\gamma(\Delta x_{i-1})}{6}[u_{i-1} + 2u_i] + \frac{c^2}{\Delta x_{i-1}}[u_i - u_{i-1}] = 0 \quad (28)$$

In general $\Delta x_i = x_{i+1} - x_i$ and $\Delta x_{i-1} = x_i - x_{i-1}$. Now let $\Delta x_i = \Delta x_+$ and $\Delta x_{i-1} = \Delta x_-$. So equations (22) and (28) for ith node of (i) and $(i-1)$ element will be

$$\alpha\frac{\Delta x_+}{6}[2\ddot{u}_i + \ddot{u}_{i+1}] + \beta\frac{\Delta x_+}{6}[2\dot{u}_i + \dot{u}_{i+1}] + \frac{\gamma(\Delta x_+)}{6}[2u_i + u_{i+1}] - \frac{c^2}{\Delta x_+}[u_{i+1} - u_i] = 0 \quad (29)$$

$$\frac{\alpha(\Delta x_-)}{6}[\ddot{u}_{i-1} + 2\ddot{u}_i] + \frac{\beta(\Delta x_-)}{6}[\dot{u}_{i-1} + 2\dot{u}_i] + \frac{\gamma(\Delta x_-)}{6}[u_{i-1} + 2u_i] + \frac{c^2}{\Delta x_-}[u_i - u_{i-1}] = 0 \quad (30)$$

Multiplying equation (29) by $\frac{6}{\alpha(\Delta x_+)}$ and equation (30) by $\frac{6}{\alpha(\Delta x_-)}$ and then adding we get

$$[\ddot{u}_{i-1} + 4\ddot{u}_i + \ddot{u}_{i+1}] + \frac{\beta}{\alpha}[\dot{u}_{i-1} + 4\dot{u}_i + \dot{u}_{i+1}] - \frac{6c^2}{\alpha(\Delta x_+)^2}[u_{i+1} - u_i] + \frac{6c^2}{\alpha(\Delta x_-)^2}[u_i - u_{i-1}] + \frac{\gamma}{\alpha}[u_{i-1} + 4u_i + u_{i+1}] = 0 \quad (31)$$

Approximation of Time Derivative: The central difference approximation for n^{th} time level is given as:

$$\ddot{u}_i = \frac{u_i^{n-1} - 2u_i^n + u_i^{n+1}}{(\Delta t)^2}$$

And

$$\dot{u}_i = \frac{u_i^{n-1} - u_i^{n+1}}{2(\Delta t)}$$

Substituting this value of time derivative in equation (31) at n^{th} level of time, we have,

$$\left[\frac{u_{i-1}^{n-1} - 2u_{i-1}^n + u_{i-1}^{n+1}}{(\Delta t)^2} + 4\frac{u_i^{n-1} - 2u_i^n + u_i^{n+1}}{(\Delta t)^2} + \frac{u_{i+1}^{n-1} - 2u_{i+1}^n + u_{i+1}^{n+1}}{(\Delta t)^2}\right] + \frac{\beta}{\alpha}\left[\frac{u_{i-1}^{n+1} - u_{i-1}^{n-1}}{2(\Delta t)} + 4\frac{u_i^{n+1} - u_i^{n-1}}{2(\Delta t)} + \frac{u_{i+1}^{n+1} - u_{i+1}^{n-1}}{2(\Delta t)}\right] - \frac{6c^2}{\alpha(\Delta x_+)^2}[u_{i+1}^n - u_i^n] + \frac{6c^2}{\alpha(\Delta x_-)^2}[u_i^n - u_{i-1}^n] + \frac{\gamma}{\alpha}[u_{i-1}^n + 4u_i^n + u_{i+1}^n] = 0$$

On simplifying we get

$$\left[1+\frac{\beta(\Delta t)}{2\alpha}\right](u_{i-1}^{n+1}+4u_i^{n+1}+u_{i+1}^{n+1}) = \left[\frac{\beta(\Delta t)}{2\alpha}-1\right](u_{i-1}^{n-1}+4u_i^{n-1}+u_{i+1}^{n-1}) + \left[2+\frac{6c^2(\Delta t)^2}{\alpha(\Delta x_-)^2}-\frac{\gamma(\Delta t)^2}{\alpha}\right]u_{i-1}^n + \left[8-\frac{6c^2(\Delta t)^2}{\alpha(\Delta x_+)^2}-\frac{6c^2(\Delta t)^2}{\alpha(\Delta x_-)^2}-\frac{4\gamma(\Delta t)^2}{\alpha}\right]u_i^n + \left[2+\frac{6c^2(\Delta t)^2}{\alpha(\Delta x_+)^2}-\frac{\gamma(\Delta t)^2}{\alpha}\right]u_{i+1}^n = 0 \quad (32)$$

Equation (32) represents the Finite Element Scheme for second order Hyperbolic Partial Differential Equations when we have non-uniform grid. Now for uniform grids we have

$$\Delta x_+ = \Delta x_- = \Delta x$$

Therefore from (32) we have

$$\left[1+\frac{\beta(\Delta t)}{2\alpha}\right](u_{i-1}^{n+1}+4u_i^{n+1}+u_{i+1}^{n+1}) = \left[\frac{\beta(\Delta t)}{2\alpha}-1\right](u_{i-1}^{n-1}+4u_i^{n-1}+u_{i+1}^{n-1}) + \left[2+\frac{6c^2(\Delta t)^2}{\alpha(\Delta x)^2}-\frac{\gamma(\Delta t)^2}{\alpha}\right]u_{i-1}^n + \left[8-\frac{12c^2(\Delta t)^2}{\alpha(\Delta x)^2}-\frac{4\gamma(\Delta t)^2}{\alpha}\right]u_i^n + \left[2+\frac{6c^2(\Delta t)^2}{\alpha(\Delta x)^2}-\frac{\gamma(\Delta t)^2}{\alpha}\right]u_{i+1}^n = 0$$

$$\left[1+\frac{\beta(\Delta t)}{2\alpha}\right](u_{i-1}^{n+1}+4u_i^{n+1}+u_{i+1}^{n+1}) = \left[\frac{\beta(\Delta t)}{2\alpha}-1\right](u_{i-1}^{n-1}+4u_i^{n-1}+u_{i+1}^{n-1}) + \left[2-\frac{\gamma(\Delta t)^2}{\alpha}\right](u_{i-1}^n+4u_i^n+u_{i+1}^n) + \left[\frac{6c^2(\Delta t)^2}{\alpha(\Delta x)^2}\right](u_{i-1}^n-2u_i^n+u_{i+1}^n) = 0$$

Let $d = \frac{6c^2(\Delta t)^2}{\alpha(\Delta x)^2}$

Therefore we have:

$$\left[1+\frac{\beta(\Delta t)}{2\alpha}\right](u_{i-1}^{n+1}+4u_i^{n+1}+u_{i+1}^{n+1}) = \left[\frac{\beta(\Delta t)}{2\alpha}-1\right](u_{i-1}^{n-1}+4u_i^{n-1}+u_{i+1}^{n-1}) + \left[2-\frac{\gamma(\Delta t)^2}{\alpha}\right](u_{i-1}^n+4u_i^n+u_{i+1}^n) + d(u_{i-1}^n-2u_i^n+u_{i+1}^n) = 0 \quad (33)$$

Equation (33) represents the Finite Element Model for linear second order one dimensional Telegraph equation with uniform mesh points.

Test Problem: Let us consider the Telegraph equation as

$$\pi^2\frac{\partial^2 u}{\partial t^2} + 3\pi^2\frac{\partial u}{\partial t} + \pi^2 u = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad t > 0$$

and initial conditions

$$\left.\begin{aligned} u(x,0) &= \sin\pi x \\ \frac{\partial u}{\partial t}(x,0) &= -\sin\pi x \end{aligned}\right\} \qquad 0 \le x \le 1$$

Exact Solution: $u(x,t) = e^{-t}\sin\pi x$

Table 1: Comparison of FEM and EXACT Solutions at h = 0.1, k = 0.02

X	FEM Solution	Exact Solution	Error
0.0	0.000000000	0.000000000	0.000000000
0.1	0.302898152	0.302898048	1.04E-07
0.2	0.576146521	0.576146324	1.97E-07
0.3	0.792997656	0.792997385	2.71E-07
0.4	0.932224654	0.932224336	3.18E-07
0.5	0.980199008	0.980198673	3.35E-07
0.6	0.932224654	0.932224336	3.18E-07
0.7	0.792997656	0.792997385	2.71E-07
0.8	0.576146521	0.576146324	1.97E-07
0.9	0.302898152	0.302898048	1.04E-07
1.0	0.000000000	0.000000000	0.000000000

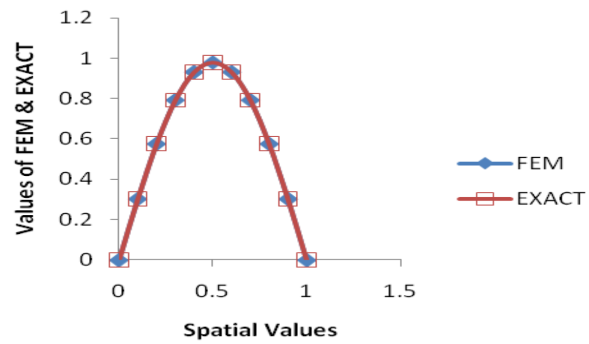

Figure 3. Comparison of FEM and EXACT

RESULTS AND DISCUSSION

In Table 1, a comparison of FEM solution with exact solution along with absolute error is presented. It can be observed that computed values are very close to exact values and corresponding error is negligible. In Figure 3, both FEM and exact solutions are plotted. It is clear from the plot that solution obtained by developed scheme is approximately equal to exact solution.

Conclusion: A Galerkin based Finite Element Model for linear second order one dimensional Telegraph equation has been developed. Accuracy of the developed scheme has been analyzed by solving a test problem and comparing computed values with exact solutions. It is observed that the scheme produces very accurate results.

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