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FINITE DIFFERENCE METHODS FOR THE SOLUTION OF KORTEWEG DE-VRIES EQUATION

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ABSTRACT: The purpose of this paper is to set up and analyze difference schemes for solving the initial value problem for well-known Korteweg de-Vries (simply K-dV) equation. We have examined four different numerical schemes of finite difference method for K-dV equation. After the discussion of a difference scheme which is correctly centered in both space and time, the construction of difference schemes which implicitly contain the effect of dissipation is described. Then we have compared the difference scheme with the exact solution of Korteweg de-Vries equation.

INTRODUCTION

The Korteweg-de Vries equation

$$U_t + UU_x + U_{xxx} = 0$$

was first derived by Korteweg and Vries (1895) to model water waves in a shallow canal, when the study of water waves was of vital interest for applications in naval architecture and for the knowledge of tides and floods. The K-dV equation did not receive further significant attention until 1965, when Zabusky and Kruskal (1965) published the results of their numerical experimentation with the equation after which this equation raised a wide spread interest. Shortly after that, a paper by Gardner, Greene, Kruskal, and Miura (1967) was a remarkable discovery.

K-dV equation has subsequently been found to be involved in a wide range of physics phenomena, especially those exhibiting shock waves, traveling waves, and solitons, (Wazwaz, 2001). Certain theoretical physics phenomena in the quantum mechanics domain are explained by means of a K-dV model. It is used in fluid dynamics, aerodynamics, and continuum mechanics as a model for shock wave formation, solitons, turbulence boundary layer behavior and mass transport.

Formulation

The Korteweg de-Vries equation is a third order non-linear partial differential equation.

$$U_t + UU_x + U_{xxx} = 0$$

subject to the periodic boundary conditions

$$U(x + 2\pi, t) = U(x, t)$$

we consider this boundary value problem along with the initial condition

$$U(x, 0) = 3\operatorname{sec}h^2 \frac{x}{2}$$

It can be easily verified that exact solution of K-dV equation is

$$U(x, t) = 3\operatorname{sec}h^2\left(\frac{x-t}{2}\right)$$

METHODS OF SOLUTION

We consider the following four different numerical schemes:

i) Second Order Finite Difference Method 1 (SOFDM 1)

ii) Second Order Finite Difference Method 2 (SOFDM 2)

iii) Fourth Order Finite Difference Method 1 (FOFDM 1)

iv) Fourth Order Finite Difference Method 2 (FOFDM 2)

Second Order Finite Difference Method 1 (SOFDM1)

In order to construct a second order finite difference method 1, we write K-dV equation as the following coupled system of PDE's

$$U_t + UV + V_{xx} = 0$$

$$U_x - V = 0$$

We then use second order central difference approximations for the first and second order spatial derivatives, KW Morton (2005), to obtain the following system of linear algebraic equations and first order non-linear differential equations:

$$V_i = \frac{U_{i+1} - U_{i-1}}{2h}$$

For i = 1, $V_1 = \dfrac{U_2 - U_n}{2h}$

For i = 2, 3… n-1

$$V_i = \frac{U_{i+1} - U_{i-1}}{2h}$$

For i = n, $V_1 = \dfrac{U_1 - U_{n-1}}{2h}$

$$\frac{dU_i}{dt} = -U_i V_i - \frac{V_{i-1} - 2V_i + V_{i+1}}{h^2}$$

Second Order Finite Difference Method 2 (SOFDM2)

In order to construct a second order finite difference method 2, we write K-dV equation as the following coupled system of PDE's

$$U_t + UU_x + V_x = 0$$

$$U_{xx} - V = 0$$

We then use second order central difference approximations for the first and second order spatial derivatives, (Morton, 2005) to obtain the following system of linear algebraic equations and first order non-linear differential equations:

$$V_i = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}$$

For i = 1, $V_1 = \dfrac{U_n - 2U_1 + U_2}{h^2}$

For i = 2 , 3 … n-1

$$V_i = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}$$

For i = n, $V_n = \dfrac{U_{n-1} - 2U_n + U_1}{h^2}$

$$\frac{dU_i}{dt} = -U_i \frac{U_{i+1} - U_{i-1}}{2h} - \frac{V_{i+1} - V_{i-1}}{2h}$$

Fourth Order Finite Difference Method1 (FOFDM1)

In order to construct a fourth order finite difference method 1, we write K-dV equation as the following coupled system of PDE's

$$U_t + UV + V_{xx} = 0$$

$$U_x - V = 0$$

We then use fourth order central difference approximations for the first and second order spatial derivatives (Morton, 2005) to obtain the following system of linear algebraic equations and first order non-linear differential equations:

$$V_i = \frac{U_{i-2} - 8U_{i-1} + 8U_{i+1} - U_{i+2}}{12h}$$

For i = 1, $V_1 = \dfrac{U_{n-1} - 8U_n + 8U_{n+1} - U_{n+2}}{12h}$

For i = 2, $V_i = \dfrac{U_n - 8U_1 + 8U_3 - U_4}{12h}$

For i = 3, n-2 $V_n = \dfrac{U_{i-2} - 8U_{i-1} + 8U_{i+1} - U_{i+2}}{12h}$

For i = n-1, $V_{n-1} = \dfrac{U_{n-3} - 8U_{n-2} + 8U_n - U_1}{12h}$

For i = n $V_n = \dfrac{U_{n-2} - 8U_{n-1} + 8U_1 - U_2}{12h}$

$$\frac{dU_i}{dt} = -U_i V_i - \left(\frac{-V_{i-2} + 16V_{i-1} - 30V_i + 16V_{i+1} - V_{i+2}}{12h^2} \right)$$

Fourth Order Finite Difference Method 2 (FOFDM2)

In order to construct a fourth order finite difference method 2, we write K-dV equation as the following coupled system of PDE's

$$U_t + UU_x + V_x = 0$$

$$U_{xx} - V = 0$$

We then use fourth order central difference approximations for the first and second order spatial derivatives (Morton, 2005) to obtain the following system of linear algebraic equations and first order non-linear differential equations:

$$V_i = \frac{U_{i-2} + 16U_{i-1} - 3U_i + 16U_{i+1} - U_{i+2}}{12h^2}$$

For i = 1, $V_1 = \dfrac{U_{n-1} + 16U_n - 3U_1 + 16U_2 - U_3}{12h^2}$

For i = 2, $V_2 = \dfrac{U_n + 16U_1 - 3U_2 + 16U_3 - U_4}{12h^2}$

For i = 3, n-2 $V_i = \dfrac{U_{i-2}+16U_{i-1}-3U_i+16U_{i+1}-U_{i+2}}{12h^2}$

For i = n-1, $V_{n-1} = \dfrac{U_{n-3}+16U_{n-2}-3U_{n-1}+16U_n-U_1}{12h^2}$

For i = n, $V_n = \dfrac{U_{n-2}+16U_{n-1}-3U_n+16U_1-U_2}{12h^2}$

$$\frac{dU_i}{dt} = -U_i\left(\frac{U_{i-2}-8U_{i-1}+8U_{i+1}-U_{i+2}}{12h}\right)-\left(\frac{V_{i-2}-8V_{i-1}+8V_{i+1}-V_{i+2}}{12h}\right)$$

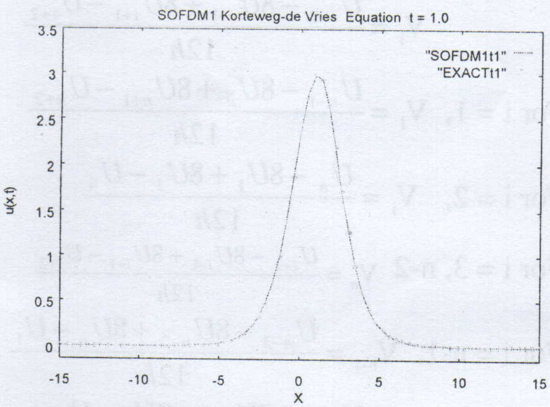


Fig.1

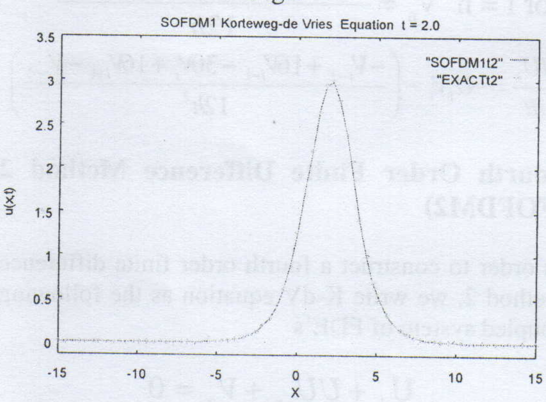


Fig.2

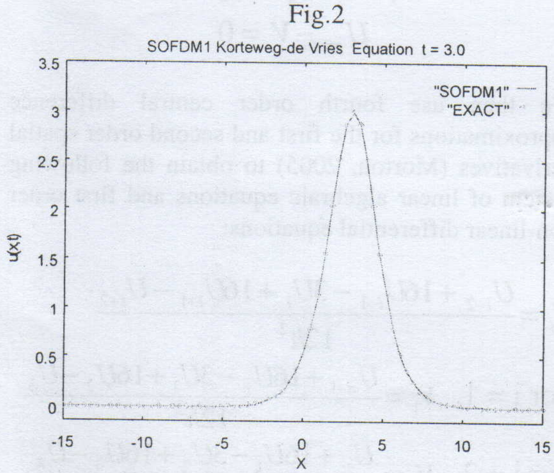


Fig.3

RESULTS AND DISCUSSION

Using the four different schemes described in the previous section, we carried out to investigate the behavior of the solution of Korteweg de-Vries equation.

The accuracy of computed solutions is examined by comparison with the exact solution of Korteweg de-Vries equation. The close results agreement between the current results and the exact solutions confirms that the proposed finite-difference procedure is an effective technique for the solution of the Korteweg de-Vries equation.

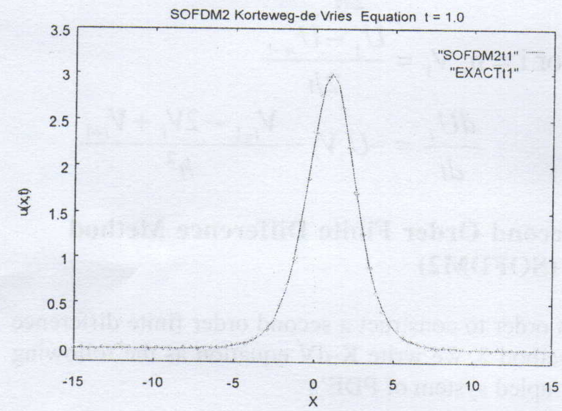


Fig.4

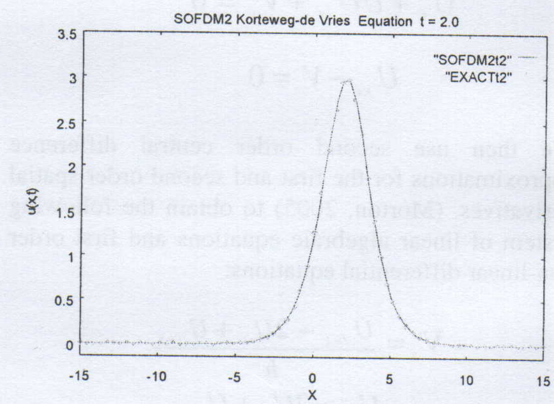


Fig.5

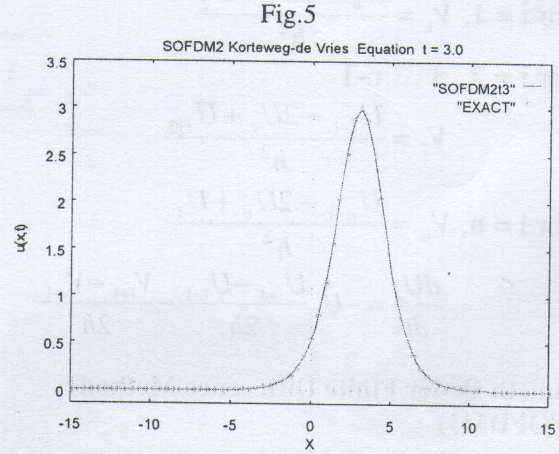
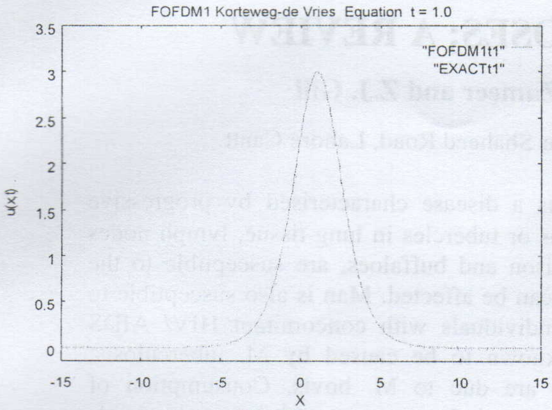


Fig.6



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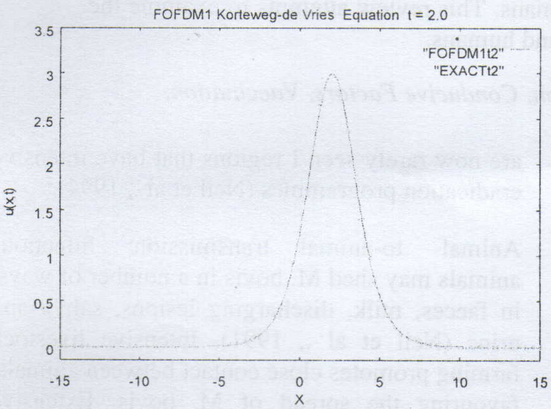

Fig 8

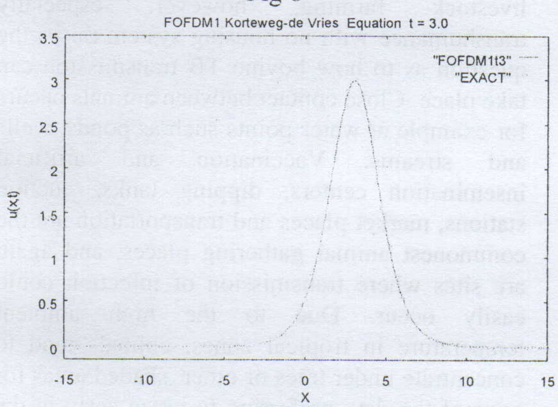

Fig 9

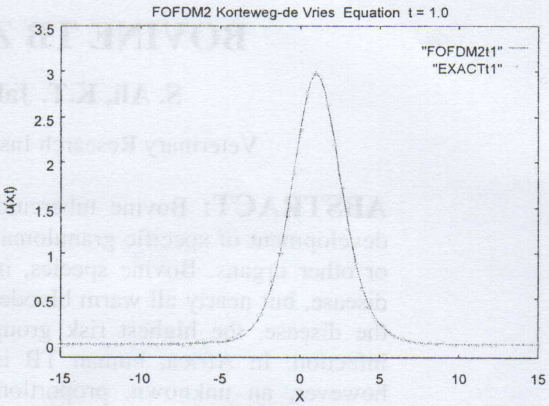

Fig 10

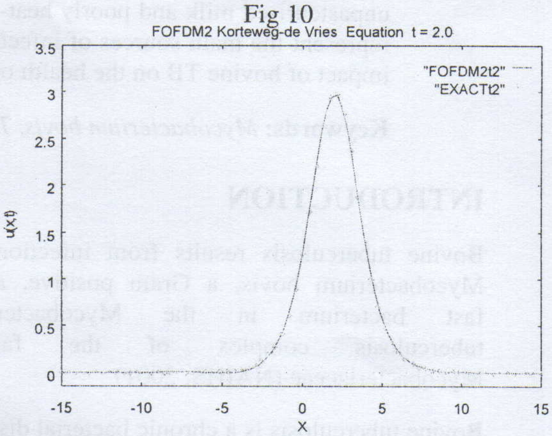

Fig 11

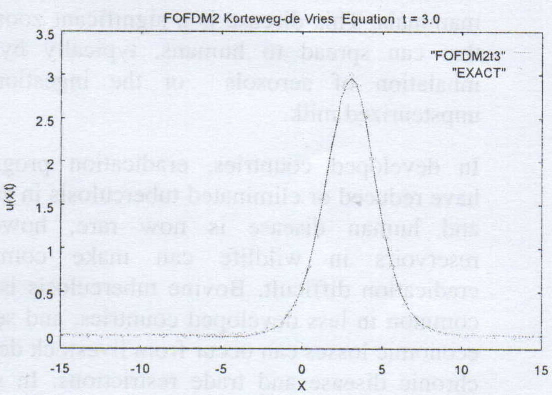

Fig 12

Figures 1-12: Dots indicate the exact solution of Korteweg de-Vries equation while lines indicate the numerical solution of Korteweg de-Vries equation using finite difference approximation.

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