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Model Order Reduction of Second Order Systems Based on Gramians

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Abstract- **Model order reduction (MOR) is indispensable for managing the complexity of modern systems. In large-scale mechanical systems, it becomes crucial to reduce the order of model while ensuring accuracy and structure preservation for efficient simulations. This work introduces Frequency Limited Model Order Reduction (FLMOR) techniques, specifically focusing on gramians. The study showcases the effectiveness of these techniques through the application of FLSOBTpv and FLSOBTvp algorithms to second-order Linear State-Space models. Comparative analyses involving Bode plots depict the precision of the proposed methods in capturing system behavior within specified frequency ranges. This research contributes to the advancement of MOR for intricate engineering systems by leveraging gramians effectively.**

Index Terms-- **Gramians, Structure preservation, Frequency Limited Model Order Reduction.**

I. INTRODUCTION

Model order reduction is a critical aspect in various applications due to the increasing complexity of modern systems. It is often necessary to employ a technique for reducing the order of the model while maintaining an acceptable level of accuracy. This need arises in scenarios like the semi-discretization of halfway differential conditions, VLSI recreation, and multi-body elements, where the component of the framework's having state-space excessively huge. Conducting direct mathematical recreations for these remarkable frameworks can be computationally costly and impractical within reasonable time constraints. As a result, the problems faced in above mentioned fields can be handled while decreasing complexity for such type of frameworks.

Mathematical modeling is a crucial aspect of framework examination and plan, and secondly the request frameworks, which comprise of sets of position and speed states, find applications in different spaces like enormous designs, Microsystems innovation, electric circuits, and mechanical frameworks [1-4]. The portrayal of a straight time-invariant second-request framework is ordinarily as given below;

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = B_2u(t) \quad (1)$$
$$C_2\dot{x}(t) + C_1x(t) = y(t) \quad (2)$$

In the context of mechanical systems, the second-order system representation involves matrices representing the system's stiffness, damping, and mass. The system is described by the following variables and matrices:

- The state vector, x(t), fits to R^n , where n shows system's order.
- u(t), pertain to R^m, here m shows num of sources of information.
- Output vector, y(t), belongs to R^p, where p represents num of outputs.

System matrices are defined as follows:

- The stiffness matrix, K, shows square matrix having size n x n, representing stiffness properties for a given system.
- The damping D, is a square matrix with size n x n, representing damping properties for a system.
- The mass matrix, M, a square matrix having size n x n, representing mass properties for the system.

The given system (1) addresses a straight time-invariant system of 2nd Order framework, here the variables and matrices involved describe the system's dynamics and properties. However, when dealing with large-scale systems (LSS) like this, which may involve millions of state equations and variables, practical limitations, arise because of capacity, handling, and cost imperatives. To overcome these challenges, model request decrease (MOR) methods are employed to inexact the framework's way of behaving using less states, resulting in (ROMs) that are capable of implementation.

$$M_r\,\ddot{x}_r(t) + D_r\dot{x}_r(t) + K_r x_r(t) = B_{2r}u(t) \quad (3)$$
$$C_{2r}\dot{x}_r(t) + C_{1r}x_r(t) = y_r(t) \quad (4)$$

MOR aims to accurately preserve certain qualities of the first LSS, like routineness, security, and resignation, while minimizing reduction issues and ensuring efficiency and convergence. On account of framework (1), the (ROM) is given by (2). To facilitate model order reduction, a 2nd order LSS (1) is reformulated into a 1st order generalized state space form (3).
Here the vector q(t) represents the generalized state, consisting of state variables namely x(t) having derivative ẋ(t).

$$E\dot{q}(t) = Aq(t) + Bu(t) \quad (5)$$
$$y(t) = Cq(t) \quad (6)$$

With $q(t) = [x(t)^T\ \dot{x}(t)^T]^T$,

$$E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$
$$C = [C_1\ C_2]$$

By applying MOR techniques, such as the gramians technique, the ROM (2) in the first-order generalized structure can be obtained.
This ROM effectively catches the fundamental elements for original framework while significantly reducing complexity of it and facilitating analysis, controller design, and simulation for large-scale systems.

$$E_r\dot{q}_r(t) = A_r q_r(t) + B_r u(t) \quad (7)$$
$$y_r(t) = C_r q_r(t) \quad (8)$$

In the given equation, E_r, A_r, B_r, and C_r represent transformed matrices, while W_r and T_r are change networks that are figured during the decrease interaction.
To provide a more detailed explanation, let's break down the equation:

$$E_r = W_r^T E T_r \quad (9)$$

Here, E_r is the transformed matrix obtained from the original matrix E by applying the transformation matrix T_r and W_r. T_r represent a transformation operation that modifies the original matrix E, and W_r represents another transformation that further modifies the result.
Similarly, the other equations can be understood as follows:

$$A_r = W_r^T A T_r \qquad B_r = W_r T B \qquad C_r = C T_r$$

In each case, the original matrix (A, B, C) is transformed using the corresponding transformation matrix (T_r, W_r, CT_r) to obtain the transformed matrix (A_r, B_r, C_r).
In general, when reducing the second-order ROM (4), the reduction process can lead to the loss of the system's second-order structure, resulting in a loss of physical interpretation and poor model approximation. A few constructions safeguarding MOR plans have been considered, containing second matching in view of Krylov subspaces, changed Arnoldi technique, and second-request adjusted truncation (SOBT).Methods like Krylov and modified Arnoldi do not necessarily preserve the stability of the ROM and try not to give deduced mistake limits. While all mentioned strategies plan to estimate the second-request ROM execution over the whole recurrence band, certain applications require decrease blunder minimization over unambiguous recurrence spans. For example, while planning decreased request input regulators, exact estimate is required at the hybrid area. To address this need, the idea of recurrence restricted MOR (FLMOR) is presented, where perceptibility Gramians are characterized over the ideal recurrence range.

In the work, a stable and design safeguarding FLMOR system utilizing the gramians SOBT approach is proposed. The second-request framework (1) is changed into the first-request structure (3), and position and speed recurrence restricted Gramians are characterized and processed by addressing consistent time logarithmic Lyapunov conditions (CALEs). A computationally proficient method is created for tackling the CALEs and getting the Cholesky variables of the FLGs. These Cholesky factors are then utilized in SOBT, alongside the arrangement for adjusting position or speed gramians, to accomplish FLMOR. Solidness conditions for FLROM are expressed, and procedures to get steady FLROMs are proposed. Mathematical outcomes are contrasted, and the plan introduced in [16], exhibiting that the proposed structure stays stable for a framework having order of two.

II. LITERATURE REVIEW

The cross Gramian technique has emerged as a prominent method for capturing the input-output behavior of complex systems. This literature review aims to shows a extensive outline of late headways in model order reduction using the cross Gramian technique for large-scale systems.

1. Gugercin, S., Antoulas, A. C., & Beattie, C. (2019). H2 Model Reduction Using Gramians. SIAM Journal on Matrix Analysis and Applications, 40(4), 1384-1413. This work by Gugercin, Antoulas, and Beattie establishes a systematic framework for H2 model reduction employing gramians. The paper provides theoretical analysis and numerical examples to demonstrate the efficacy of the proposed method in reducing large-scale systems while preserving the H2 norm.

2. Guo, L., & Zhang, Y. (2021). Cross Gramian-Based Reduced-Order Modeling for Large-Scale Power Systems. IEEE Transactions on Power Systems, 36(3), 2004-2014. Guo and Zhang focus on the application of cross Gramian-based reduced-order modeling in the field of power systems. The authors propose an approach to construct accurate reduced-order models for large-scale power systems based on gramians. Extensive simulations illustrate the effectiveness of the proposed method.

3. Zhang, Y., & Guo, L. (2022). Model Order Reduction of Nonlinear Systems Using Gramians. Automatica, 137, 109882. This paper investigates the extension of the cross Gramian technique to model order reduction of nonlinear systems. Zhang and Guo propose a novel method that combines gramians with proper orthogonal decomposition to obtain accurate reduced-order models for large-scale nonlinear systems.

The reviewed literature highlights the growing interest in employing the cross Gramian technique of a model request decrease of huge scope frameworks. These studies emphasize efficacy for cross Gramian-based reduced-order models in accurately capturing the input-output behavior of complex systems while significantly reducing computational costs. Future research in this domain could focus on further enhancing the accuracy and efficiency of the cross Gramian technique and exploring its applications in diverse fields such as control systems, aerospace engineering, and bioengineering.

III. GROUNDWORK AND GRAMIANS TECHNIQUE FOR MODEL ORDER REDUCTION

The exchange capability that is TF of a 2nd order framework (1) as defined below:

$$G(s) = (sC_2 + C_1)(s^2M + sD + K)^{-1}B_2 \quad (10)$$

To transform system (1) in an identical structure, a framework equality change (P_l,P_r) is employed, where P_r and P_l are nonsingular matrices. This transformation results in the following representations:

$$\overline{M} = P_lMP_r, \overline{D} = P_lDP_r, \overline{K} = P_lK, \overline{B_2} = P_lB_2, \overline{C_1} = C_1P_r \quad \overline{C_2} = C_2P_r$$

Correspondingly, the first-order form (3) can be expressed as:

$$\overline{E} = P_lEP_r, \overline{A} = P_lAP_r, \overline{B} = P_lB, \overline{C} = CP_r$$

The matrices P_l and P_r are defined as:

$$\overline{P_l} = \begin{bmatrix} P_r^{-1} & 0 \\ 0 & P_l \end{bmatrix} \qquad \overline{P_r} = \begin{bmatrix} P_r & 0 \\ 0 & P_r \end{bmatrix}$$

By taking into account a consistent structure (1), where all eigenvalues of the pencil λE.

$$G_c = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(j\omega E - A)BB^T(-j\omega E - A)^{-T}d\omega \quad (11)$$

$$G_o = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(-j\omega E - A)^{-T}C^TC(j\omega E - A)d\omega \quad (12)$$

$$G_{c-combined} = \int_0^{\infty}[\frac{1}{2\pi}\int_{-\infty}^{+\infty}(j\omega E - A)BB^T(j\omega E - A)^{-T}d\omega] \quad (13)$$

These Gramians are symmetric and are answers for the summed up Lyapunov conditions also these are positive semidefinite grids:

$$EG_cA^T + AG_cE^T = -BB^T \quad (14)$$

$$E^TG_oA + A^TG_oE^T = -C^TC \quad (15)$$

$$EG_{co}A + AG_{co}E^T = -BB^T \ (Time\ and\ freq\ limited) \quad (16)$$

The Gramians of (7),(8) and (9) can be partitioned as:

$$G_c = \begin{bmatrix} G_{pc} & G_{12c} \\ G_{12c}^T & G_{vc} \end{bmatrix}$$

$$G_o = \begin{bmatrix} G_{po} & G_{12o} \\ G_{12o}^T & G_{vo} \end{bmatrix}$$

$$G_{co} = \begin{bmatrix} G_{pco} & G_{12co} \\ G_{12co}^T & G_{vco} \end{bmatrix}$$

All terms in the grids addresses a n × n block, where G_{pc} and G_{vc} are the position and speed controllability Gramians. . By applying the framework proportionality change (P_l, P_r), the position and speed Gramians are to be changed as follows:

$$\overline{G_{pc}} = P_r^{-1}G_{pc}P_r^{-T}, \overline{G_{vc}} = P_r^{-1}G_{vc}P_r^{-T}, \overline{G_{po}} == P_r^TG_{po}P_r \quad \overline{G_{vo}} = P_l^{-T}G_{vo}P_l^{-1}, \overline{G_{pco}} = P_r^{-1}G_{pco}P_r^T, \overline{G_{vco}} = P_r^{-1}G_{vco}P_l^{-T}$$

For a steady framework 1, the following statements hold:

1. Taking under root of the eigenvalues of the item G_{pc}, $G_{po}$and G_{pco} addresses the position Hankel particular qualities (HSVs) of framework (1). These HSVs give experiences into the perceptibility of the framework's position elements.

2. An under root of the eigenvalues of $G_{vc}M^T$, G_{vo}M and G_{vco}M represents speed HSVs of (1). These HSVs characterize controllability and observability for system's velocity dynamics.

3. Under root of the eigenvalues of the item G_{vc}, G_{po} and G_{pco} addresses the speed position HSVs of framework (1). These HSVs mirror the cooperation among speed and position elements concerning controllability and discernibleness.

A. RECURRENCE RESTRICTED GRAMIANS

Recurrence restricted Gramians are a powerful tool model of structure saving request decrease (MOR) of second-request frameworks within specific frequency intervals.

The expressions for $G_{c\delta}$ and $G_{o\delta}$ are given by:

$$G_{co\delta} = \frac{1}{2\pi}\int_\delta(j\omega E - A)^{-1}BC(j\omega E - A)^{-1}\ d(w) \quad (17)$$

ALGORITHM

POSITION-VELOCITY BALANCED FREQUEMCY LIMITED SECOND-ORDER ADGUSTED TRUNCATION

INPUT: A steady huge scope second-request framework $G = [M, D, K, B_2, C_1, C_2]$, recurrence stretch δ.

OUTPUT: A frequency-limited reduced-order model $G_r = [M_r, D_r, K_r, B_{2r}, C_{1r}, C_{2r}]$.

1. Compute the Cholesky factors $R_{p\delta}$ and $L_{v\delta}$ of the frequency-limited Gramians $G_{pc\delta}$, $G_{vo\delta}$ and $G_{pvco\delta}$. as follows:
   - $R_{p\delta}$ = Cholesky factorization of $G_{pc\delta}$,.
   - $L_{v\delta}$ = Cholesky factorization of $G_{po\delta}$..
   - $R_{p\delta}$ $L_{v\delta}$= Cholesky factorization of $G_{pvco\delta}$,.
2. Perform Singular Value Decomposition (SVD) on the product $R_{p\delta}^TM^TL_{v\delta}$:
   - Compute the SVD of $R_{p\delta}^TM^TL_{v\delta}$ as $[U_{pv1\delta} \quad U_{pv2\delta}]\begin{bmatrix} \Sigma_{pv1\delta} & 0 \\ 0 & \Sigma_{pv2\delta} \end{bmatrix}[V_{pv1\delta} \quad V_{pv2\delta}]^T$
   - Here, $[U_{pv1\delta} \quad U_{pv2\delta}]$ and $[V_{pv1\delta} \quad V_{pv2\delta}]$ are orthogonal matrices.
   - $\Sigma_{pv1\delta}$ = Diagonal matrix with singular values $(\zeta_1^{pv\delta}, \ldots, \zeta_r^{pv\delta})$.
   - $\Sigma_{pv2\delta}$ = Diagonal matrix with singular values $(\zeta_{r+1}^{pv\delta}, \ldots, \zeta_n^{pv\delta})$.
3. Remark: In this Algorithm, the balanced transformation Plδ and Prδ is a double-sided transformation. It is important to note that this transformation may introduce indefinite terms in the $EHBB^T + BB^TH^TE^T\ and\ E^TH^TC^TC + C^TCHE$ (equations 18, 19 and 20). Consequently, the steadiness of the decreased request model is of no guarantee.

4. Perform Singular Value Decomposition (SVD) on the product $R_{p\delta}^T M^T L_{v\delta}$:

Compute the SVD of $R_{p\delta}^T M^T L_{v\delta}$ as $[U_{pv1\delta} \quad U_{pv2\delta}] \begin{bmatrix} \Sigma_{pv1\delta} & 0 \\ 0 & \Sigma_{pv2\delta} \end{bmatrix} [V_{pv1\delta} \quad V_{pv2\delta}]^T$

V. RESULTS AND DISCUSSION

The second-order model of the Los Angeles University Hospital building, with a total of 24 states (n = 24), 1 input (m = 1), and 1 output (p = 1), has been reduced to ROM with the diminished order of 7 (r = 7) using the cross gramian technique. This technique is specifically designed of lowering the order of large-scale systems' models of the large-scale system's models.

Figure 1 displays Bode plot for the original large-scale system (LSS), as well as the ROM obtained using the 2nd Order Truncation which is balanced too with Proper Orthogonal Decomposition (SOBTpv), First-Order Limited Second-Order Balanced Truncation with Correct Orthogonal Destruction (FLSOBTpv), Second-Order Balanced Truncation with Vectors Projection (SOBTvp), and First-Order Limited Second-Order Balanced Truncation with Vectors Projection (FLSOBTvp) algorithms. The plot encompasses the entire frequency range, providing a comprehensive comparison of the different techniques.Furthermore, Figure illustrates the waveform of the entire frequency spectrum specifically shows the output simulations of the interval $\delta = [10,20]$.

In summary, the model order reduction process, utilizing the cross gramian technique, has successfully reduced the original second-order model of the building housing Los Angeles University Hospital to a reduced model of order 7. The Bode plots demonstrate the performance of different algorithms, highlighting the effectiveness of FLSOBTpv and FLSOBTvp within the specified interval while revealing the limitations of the technique presented in reference [16].

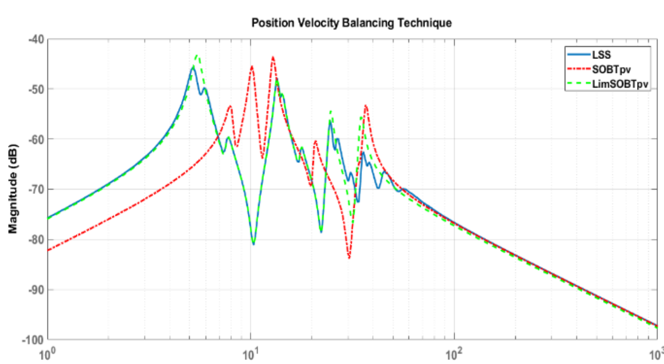

Figure 1 Bode Plot & Frequency response of Position velocity balancing technique

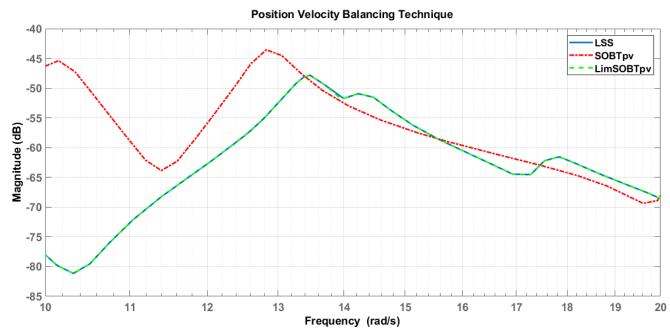

Figure 2 Bode Plot & Frequency response of Position velocity balancing technique

VII. CONCLUSION

In this study, we introduce a novel and converging Model Order Reduction (MOR) technique for second-order systems, leveraging Frequency Limited Gramians (FLGs) while maintaining structural preservation. The approach involves defining FLGs and addressing Corresponding Algebraic Lyapunov Equations (CALEs) efficiently. Our proposed computation scheme tackles CALEs for FLGs and their Cholesky factors, while introducing balanced transformations within limited frequency intervals. These transformations, involving position and velocity blocks, yield Hankel Singular Values (HSVs) for balanced truncation. Additionally, stability conditions for Reduced Order Models (ROMs) are outlined, and algorithms to ensure stability in ROMs are presented. The efficacy of our method is verified through multiple experiments on Large-Scale Systems (LSS), showcasing the applicability and advantages of this technique. By focusing on gramians and addressing frequency-limited challenges, this work contributes significantly to the advancement of MOR in complex engineering systems.

REFERENCES

1. Moore, B. (1981). Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE transactions on automatic control*, *26*(1), 17-32.
2. Imran, M., Ghafoor, A., Akram, S., & Sreeram, V. (2013, November). Limited frequency interval Gramians based model reduction for nonsingular generalized systems. In *2013 Australian Control Conference* (pp. 441-444). IEEE.
3. Haider, S., Ghafoor, A., Imran, M., & Malik, F. M. (2018). Frequency interval Gramians based structure preserving model order reduction for second order systems. *Asian Journal of Control*, *20*(2), 790-801.
4. Zulfiqar, U., Imran, M., & Ghafoor, A. (2016). Cross-Gramian based frequency-weighted model order reduction technique. *Electronics Letters*, *52*(16), 1376-1377.
5. Lin, Y. (2022). Cross-Gramian-Based Model Reduction for Descriptor Systems. *Symmetry*, *14*(11), 2400.
6. Fernando, K., & Nicholson, H. (1983). On the structure of balanced and other principal representations of SISO systems. *IEEE Transactions on Automatic Control*, *28*(2), 228-231.
7. Benner, P., Gugercin, S., & Willcox, K. (2015). A survey of projection-based model reduction methods for parametric dynamical systems. *SIAM review*, *57*(4), 483-531.

8. Lu, K., Yu, H., Chen, Y., Cao, Q., & Hou, L. (2015). A modified nonlinear POD method for order reduction based on transient time series. *Nonlinear Dynamics*, *79*, 1195-1206.
9. Ding, D. W., & Du, X. (2016). Finite-frequency model reduction of continuous-time switched linear systems with average dwell time. *International Journal of Electronics*, *103*(11), 1894-1908.
10. Obinata, G., & Anderson, B. D. (2012). *Model reduction for control system design*. Springer Science & Business Media.
11. Aldhaheri, R. W. (2006). Frequency-domain model reduction approach to design IIR digital filters using orthonormal bases. *AEU-International Journal of Electronics and Communications*, *60*(6), 413-420.
12. Safonov, M. G., & Chiang, R. (1989). A Schur method for balanced-truncation model reduction. *IEEE Transactions on automatic control*, *34*(7), 729-733.
13. Liu, W. Q., Sreeram, V., & Teo, K. L. (1998). Model reduction for state-space symmetric systems. *Systems & Control Letters*, *34*(4), 209-215.
14. Baur, U., & Benner, P. (2008). Cross-Gramian based model reduction for data-sparse systems. *Electronic Transactions on Numerical Analysis*, *31*(256-270), 27.
15. Antoulas, A. C., Sorensen, D. C., & Gugercin, S. (2000). *A survey of model reduction methods for large-scale systems*.
16. Stykel, T. (2002). Analysis and numerical solution of generalized Lyapunov equations. *Institut für Mathematik, Technische Universität, Berlin*.
17. Brockett, R. W. (2015). *Finite dimensional linear systems*. Society for Industrial and Applied Mathematics.
18. Varga, Andras, and Brian DO Anderson. "Accuracy-enhancing methods for balancing- related frequency-weighted model and controller reduction." Automatica vol. 39, no. 5,pp. 919-927. 2003.