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CERTAIN H -SUBSETS OF $Q(\sqrt{m})\setminus Q$ UNDER THE ACTION OF $H = \langle x, y : x^2 = y^4 = 1\rangle$

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Let $C' = C \cup \{\infty\}$ be the extended complex plane and $H = \langle x, y : x^2 = y^4 = 1\rangle$, where $x(z) = \frac{-1}{2z}$ and $y(z) = \frac{-1}{2(z+1)}$ are the linear fractional transformations from $C' \to C'$. Let m be a square-free positive integer. Then $Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c \neq 0, b = \frac{a^2-n}{c} \in Z \text{ and } (a,b,c) = 1\}$ where $n = k^2 m$, is a proper subset of $Q(\sqrt{m})$ for all $k \in N$. For non-square $n = 2^h \prod_{i=1}^{r} p_i^{k_i}$, it was proved in an earlier paper that the set $Q''(\sqrt{n}) = \{\frac{\alpha}{t} : \alpha \in Q^*(\sqrt{n}), t = 1, 2\}$ is an H -set for all $h \geq 0$ whereas if $h = 0$ or 1 then $Q^{**}\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}), 2 \mid c\}$ and $Q^{*:}(\sqrt{4n}) = \left(Q^*(\sqrt{n}) \setminus Q^{**}(\sqrt{n})\right) \cup Q^{**}(\sqrt{4n})$ are disjoint H -subsets of $Q''(\sqrt{n}) = Q^{**}(\sqrt{n}) \cup Q^{*:}(\sqrt{4n})$. In this paper, we prove that if $h \geq 2$, then $Q''(\sqrt{n}) = Q^{*:}(\sqrt{n}) \cup Q^{*:}(\sqrt{4n})$ and also determine the proper H -subsets of $Q^{*:}(\sqrt{4n})$. In particular, $Q(\sqrt{m}) \setminus Q = \cup Q''(\sqrt{k^2 m})$ for all $k \in N$. AMS Mathematics subject classification (2000): 05C25, 11E04, 20G15

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INTRODUCTION

Throughout the paper we take m as a square free positive integer. Since every element of $Q(\sqrt{m})\setminus Q$ can be expressed uniquely as $\frac{a+\sqrt{n}}{c}$, where $n = k^2 m$, k is any positive integer and $a, b = \frac{a^2-n}{c}$ and c are relatively prime integers and we denote it by $\alpha(a,b,c)$. Then

$$Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c, b = \frac{a^2-n}{c} \in Z \text{ and } (a,b,c) = 1\},$$

$$Q''(\sqrt{n}) = \{\frac{\alpha}{t} : \alpha \in Q^*(\sqrt{n}), t = 1, 2\},$$

$$Q^{**}\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}) \text{ and } 2 \mid c\}$$

are subsets of the real quadratic field $Q(\sqrt{m})$ and $Q(\sqrt{m})\setminus Q$ is the disjoint union of $Q^*(\sqrt{n})$ for all n.

If $\alpha(a,b,c) \in Q^*(\sqrt{n})$ and its conjugate $\bar{\alpha}$ have opposite signs then α is called an ambiguous number (Mushtaq, 1988). A non-empty set Ω with an action of a group G on it, is said to be a G -set. We say that Ω is a transitive G -set if, for any p, q in Ω there exists a g in G such that $p^g = q$.

We are interested in linear-fractional transformations x, y satisfying the relations $x^2 = y^r = 1$, with a view to study an action of the group $\langle x, y\rangle$ on real quadratic fields. If $y : z \to \frac{az+b}{cz+d}$ is to act on all real quadratic fields then a,b,c,d must be rational numbers, and can be taken to be integers. Thus $\frac{(a+b)^2}{ad-bc}$ is rational. But if $z \to \frac{az+b}{cz+d}$ is of order of r, one must have $\frac{(a+b)^2}{ad-bc} = \omega + \omega^{-1} + 2$, where ω is a primitive r-th root of unity. Now $\omega + \omega^{-1}$ is rational for a primitive r-th root only if $r = 1,2,3,4\, or\, 6$, so that these are the only possible orders of y. The group $\langle x, y : x^2 = y^r = 1\rangle$ is cyclic of order 2 or D_∞ (an infinite dihedral group) according as $r = 1\, or\, 2$. For $r = 3$, the group $\langle x, y\rangle$ is the modular group $PSL(2, Z)$. The fractional linear transformations x, y with $x(z) = \frac{-1}{2z}$ and $y(z) = \frac{-1}{2(z+1)}$ generate a subgroup H of the modular group which is isomorphic to the abstract group $\langle x, y : x^2 = y^4 = 1\rangle$. It is a standard example from the theory of the modular group. The action of H on the rational projective line $Q \cup \{\infty\}$ is transitive (see Mushtaq *et al*., 1997).

In our case, the set $Q(\sqrt{m}) \setminus Q$ is an H-set. It is noted that H is the free product of $C_2 = \langle x : x^2 = 1\rangle$ and $C_4 = \langle x : y^4 = 1\rangle$. The action of the modular group $PSL(2, Z)$ on the real quadratic fields has been discussed in detail in (Mushtaq, 1988) and (M. Aslam Malik *et al*., 2005). The actual number of ambiguous numbers in $Q^*(\sqrt{n})$ has been discussed in (S. M. Husnine *et al*., 2005) as a function of n.

In a recent paper M. Aslam Malik and M. Asim Zafar, 2011, have investigated that the cardinality of the set E_p, p a prime factor of n, consisting of all classes $[a,b,c](mod\ p)$ of the elements of $Q^*(\sqrt{n})$ is $p^3 - 1$ and obtained two proper G-subsets of $Q^*(\sqrt{n})$ corresponding to each odd prime divisor of n. M. Aslam Malik and M. Asim Zafar, 2011 have determined the cardinality of the set E_{p^r}, $r \ge 1$, consisting of all classes $[a,b,c](mod\ p^r)$ of the elements of $Q^*(\sqrt{n})$ and have determined, for each non-square n, the G-subsets of an invariant subset $Q^*(\sqrt{n})$ of $Q(\sqrt{m}) \setminus Q$ under the modular group action by using classes $[a,b,c](mod\ n)$.

In this paper we examine the action of the group H on subsets $Q''(\sqrt{n})$ of $Q(\sqrt{m}) \setminus Q$. An action of H and its proper subgroup on $Q(\sqrt{m})$ has been discussed in (Mushtaq *et al*., 1993, 1997, 2007). M. Aslam Malik *et al*., 2005, examined some properties of real quadratic irrational numbers under the action of H and found some H-subsets of $Q(\sqrt{m})$. In Lemma 1.1 of (M. Aslam Malik *et al*., 2005) such properties were discussed for $n \equiv 1, 2\ and\ 3 (mod\ 4)$ and prove that $Q''(\sqrt{n})$ is the disjoint union of $Q^{**}(\sqrt{n})$ and $Q^{*:}(\sqrt{4n}) = \left(Q^*(\sqrt{n}) \setminus Q^{**}(\sqrt{n})\right) \cup Q^{**}(\sqrt{4n})$

In this paper we extend this result to all non-square $n \equiv 0 (mod\ 4)$ and show that $Q''(\sqrt{n})$ is the disjoint union of H-subsets $Q^{*:}(\sqrt{n})$ and $Q^{*:}(\sqrt{4n})$. This reveals that $Q(\sqrt{m}) \setminus Q$ is the union of $Q''(\sqrt{k^2 m})$ $\forall k \in N$. However if n and n' are two distinct non-square positive integers then $Q^*(\sqrt{n}) \cap Q^*(\sqrt{n'}) = \phi$ whereas $Q''(\sqrt{n}) \cap Q''(\sqrt{n'})$ may not be empty. In particular $Q''(\sqrt{n}) \cap Q''(\sqrt{4n}) = Q^{*:}(\sqrt{n})$ for each non-square positive integer n. In fact we prove that a superset namely

$$Q^{**}(\sqrt{4n}) \cup \{\frac{\alpha}{2} : \alpha = \frac{2a + \sqrt{4n}}{c} \in Q^*(\sqrt{4n}) \setminus Q^{**}(\sqrt{4n})\}$$

of $Q^{**}(\sqrt{4n})$, is an H-subset of $Q(\sqrt{m}) \setminus Q$. We have also found H-subsets of $Q^{*:}(\sqrt{4n})$ such that these may are may not be transitive however they will help in determining the transitive H-subsets (H-orbits) of $Q(\sqrt{m}) \setminus Q$.

The notation is standard and we follow (M. Aslam Malik *et al*. 2005, 37(2005)), (M. Aslam Malik and M. Asim Zafar, 2011) and (M. Aslam Malik and M. Asim Zafar, 2011 submitted). In particular $(\cdot / \cdot)$ denotes the Legendre symbol and $x(Y) = \{\frac{-1}{2\alpha} : \alpha \in Y\}$

for each subset Y of $Q(\sqrt{m})\setminus Q$.

Preliminaries:

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ with $b = \dfrac{a^2-n}{c}$. We tabulate the actions on $\alpha(a,b,c)$ of x, y and their combinations y^2, xy, yx and y^2x in the following table for later reference.

Table 1: The action of elements of H on $\alpha \in Q^{''}(\sqrt{n})$

α	a	b	c
$x(\alpha)$	$-a$	$\frac{c}{2}$	$2b$
$y(\alpha)$	$-a-c$	$\frac{c}{2}$	$2(2a+b+c)$
$y^2(\alpha)$	$-3a-2b-c$	$2a+b+c$	$4a+4b+c$
$xy(\alpha)$	$a+c$	$2a+b+c$	c
$yx(\alpha)$	$a-2b$	b	$-4a+4b+c$
$y^2x(\alpha)$	$3a-2b-c$	$\frac{-4a+4b+c}{2}$	$2(-2a+b+c)$

We list the following results from (M. Aslam Malik *et al*. 2005, 37(2005)), (M. Aslam Malik and M. Asim Zafar, 2011) and (M. Aslam Malik and M. Asim Zafar, 2011 submitted) for later reference.

Lemma 2.1: (M. Aslam Malik *et al*., 2005) Let $\alpha(a,b,c) \in Q^*(\sqrt{n})$. Then:

1. If $n \not\equiv 0 (mod\ 4)$ then $\frac{\alpha}{2} \in Q^{**}(\sqrt{n})$ if and only if $2 \mid b$.

2. $\frac{\alpha}{2} \in Q^{**}(\sqrt{4n})$ if and only if $2 \nmid b$.

Theorem 2.2 (M. Aslam Malik *et al*., 2005) The set $Q^{''}(\sqrt{n}) = \{\frac{\alpha}{t} : \alpha \in Q^*(\sqrt{n}), t = 1, 2\}$, is invariant under the action of H.

Theorem 2.3 (M. Aslam Malik *et al*., 2005) For each non square positive integer $n \equiv 1, 2\ or\ 3 (mod\ 4)$, $Q^{**}\sqrt{n}) = \{\alpha(a,b,c) : \alpha \in Q^*(\sqrt{n})\ and\ 2 \mid c\}$ is an H-subset of $Q^{''}(\sqrt{n})$.

It is well known that $G = \langle x, y : x^2 = y^3 = 1\rangle$ represents the modular group, where $x(z) = \frac{-1}{z}, y(z) = \frac{z-1}{z}$ are linear fractional transformations.

Theorem 2.4 (M. Aslam Malik *et al*., 2005 PUJM) If $n \equiv 0\ or\ 3 (mod\ 4)$, then $S = \{\alpha \in Q^*(\sqrt{n}) : b\ or\ c \equiv 1 (mod\ 4)\}$ and $-S = \{\alpha \in Q^*(\sqrt{n}) : b\ or\ c \equiv -1 (mod\ 4)\}$ are exactly two disjoint G-subsets of $Q^*(\sqrt{n})$ depending upon classes $[a,b,c]$ modulo 4.

Corollary 2.5 (M. Aslam Malik *et al*., 2005 PUJM) If $n \equiv 1\ or\ 2 (mod\ 4)$, then S and $-S$, as defined in Theorem 2.4, are not disjoint. W

Theorem 2.6 (M. Aslam Malik and M. Asim Zafar, 2011) Let p be an odd prime factor of n. Then both of $S_1^p = \{\alpha \in Q^*(\sqrt{n}) : (b/p)\ or\ (c/p) = 1\}$ and $S_2^p = \{\alpha \in Q^*(\sqrt{n}) : (b/p)\ or\ (c/p) = -1\}$ are G-subsets of $Q^*(\sqrt{n})$. In particular,these are the only G-subsets of $Q^*(\sqrt{n})$ depending upon classes $[a,b,c]$ modulo p.

Theorem 2.7 (M. Aslam Malik and M. Asim Zafar, 2011 submitted) Let p_1 and p_2 be distinct odd primes factors of n. Then $S_{1,1} = S_1^{p_1} \cap S_1^{p_2}$, $S_{1,2} = S_1^{p_1} \cap S_2^{p_2}$, $S_{2,1} = S_2^{p_1} \cap S_1^{p_2}$ and $S_{2,2} = S_2^{p_1} \cap S_2^{p_2}$ are four G-subsets of $Q^*(\sqrt{n})$. More precisely these are the only four G-subsets of $Q^*(\sqrt{n})$ depending upon classes $[a,b,c]$ modulo p_1p_2.

Notation: The four G-subsets defined in Theorem 2.7 can be briefly written as $S_{1\le i_1,i_2\le 2}$. More generally if n involves r distinct odd prime factors $p_1, p_2,..., p_r$, then $Q^*(\sqrt{n})$ is the disjoint union of 2^r subsets $S_{1\le i_1,i_2,i_3,...,i_r\le 2}$ which are invariant under the action of G.

The following theorem extends Theorem 2.7 for all non-square positive integers n.

Theorem 2.8 (M. Aslam Malik and M. Asim Zafar, 2011 submitted) Let $n = 2^k p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ where $p_1, p_2,..., p_r$ are distinct odd primes such that n is not equal to a single prime congruent to 1 modulo 8. Then the number of G-subsets of $Q^*(\sqrt{n})$ is 2^r namely $S_{1\le i_1,i_2,i_3,...,i_r\le 2}$ if $k = 0\, or\, 1$. Moreover if $k \ge 2$, then each G-subset X of these G-subsets further splits into two proper G-subsets $\{\alpha \in X : b\, or\, c \equiv 1(mod\ 4)\}$ and $\{\alpha \in X : b\, or\, c \equiv -1(mod\ 4)\}$. Thus the number of G-subsets of $Q^*(\sqrt{n})$ is 2^{r+1} if $k \ge 2$. More precisely these are the only G-subsets of $Q^*(\sqrt{n})$ depending upon classes $[a,b,c]$ modulo n.

3 Action of $H = \langle x, y : x^2 = y^4 = 1\rangle$ on $Q^{*:}(\sqrt{4n})$

In this section we establish that if n contains r distinct prime factors then $Q^{*:}(\sqrt{4n})$ is the disjoint union of 2^r subsets which are invariant under the action of H. However these H invariant subsets may further split into transitive H-subsets (H-orbits) of $Q^{*:}(\sqrt{4n})$, for example $Q^{*:}(\sqrt{4\cdot 37})$ splits into six orbits namely $(\sqrt{37})^H$, $(-\sqrt{37})^H$, $(\frac{1+\sqrt{37}}{3})^H$, $(\frac{1+\sqrt{37}}{-3})^H$, $(\frac{-1+\sqrt{37}}{3})^H$ and $(\frac{-1+\sqrt{37}}{-3})^H$. All these orbits are contained in $A_1^p \cup x(A_1^p)$.

Lemma 3.1 Let $n \equiv 1, 2\, or\, 3(mod\ 4)$. Let Y be any G-subset of $Q^{*:}(\sqrt{4n})$. Then $Y \cup x(Y)$ is an H-subset of $Q^{*:}(\sqrt{4n})$.

Proof: By Theorem 2.3, we know that $Q''(\sqrt{n}) \setminus Q^{**}(\sqrt{n})$ is an H-set. For any $\alpha \in Q^{*:}(\sqrt{4n})$, proof follows from the equations $x(\alpha) = \frac{-1}{2\alpha}$, $x(\frac{-1}{2\alpha}) = \alpha$, $y(\alpha) = \frac{-1}{2(\alpha+1)} = \frac{-1}{2\alpha'}$, where $\alpha' = \alpha + 1$ and $y(\frac{-1}{2\alpha}) = \frac{-1}{2\beta}$, where $\beta = \frac{-1}{2\alpha} + 1$. Since every element of the group $H = \langle x, y : x^2 = y^4 = 1\rangle$ is a word in the generators x, y of the group H and the transformations $\alpha \mapsto \alpha + 1$, $\alpha \mapsto \alpha - 1$ belong to both of the groups G and H. W

Theorem 3.2 Let $n \equiv 1, 2\, or\, 3(mod\ 4)$ be divisible by an odd prime p. Let $A_1^p = S_1^p \setminus Q^{**}(\sqrt{n})$ and $A_2^p = S_2^p \setminus Q^{**}(\sqrt{n})$. Then both $A_1^p \cup x(A_1^p)$ and $A_2^p \cup x(A_2^p)$ are H-subsets of $Q^{*:}(\sqrt{4n})$. Consequently the action of H on $Q^{*:}(\sqrt{4n})$ is intransitive.

Proof: follows from Theorem 2.6 and Lemma 3.1. W

Now we extend Theorem 3.2 for each non-square n.

Theorem 3.3 Let $n = 2^k p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_1, p_2,..., p_r$ are distinct odd primes and $k = 0\, or\, 1$. Let $A_{1\le i_1,i_2,i_3,...,i_r\le 2} = S_{1\le i_1,i_2,i_3,...,i_r\le 2} \setminus Q^{**}(\sqrt{n})$. Then $Q^{*:}(\sqrt{4n})$ is the disjoint union of 2^r subsets $A_{1\le i_1,i_2,i_3,...,i_r\le 2} \cup x(A_{1\le i_1,i_2,i_3,...,i_r\le 2})$ which are invariant under the action of H. More precisely these are the only H-subsets of $Q^{*:}(\sqrt{4n})$ depending upon classes $[a,b,c]$ modulo n.

Proof: Proof follows from Theorem 2.8 and Lemma 3.1.

W

Theorem 3.4 Let $n = 2^k p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_1, p_2, ..., p_r$ are distinct odd primes and $k \geq 2$. If S is any of the G-subsets given in Theorem 2.8 and $A = S \setminus Q^{**}(\sqrt{n})$, then $A \cup x(A)$ is an H-subset of $Q^{*:}(\sqrt{4n})$. More precisely these are the only H-subsets of $Q^{*:}(\sqrt{4n})$ depending upon classes $[a,b,c]$ modulo n.

Proof: follows directly from Theorem 2.8 and Lemma 3.1. W

If $n \equiv 0 \; or \; 3 (mod \; 4)$, then by Theorem 2.4, S and $-S$ are G-subsets of $Q^*(\sqrt{n})$ and hence by Lemma 3.1, $S \cup x(S)$ and $-S \cup x(-S)$ are distinct H-subsets of $Q''(\sqrt{n})$. Whereas if $n \equiv 1 \; or \; 2 (mod \; 4)$, then by Corollary 2.5, we know that S and $-S$ are not G-subsets of $Q^*(\sqrt{n})$. However the following lemma shows that $S \cup x(S)$ and $-S \cup x(-S)$ are distinct H-subsets of $Q''(\sqrt{n})$.

Lemma 3.5 Let $X = Y \setminus Q^{**}(\sqrt{n})$, where Y is any of the G-subsets of $Q^*(\sqrt{n})$ and $n \equiv 1 \; or \; 2 (mod \; 4)$. Let $S = \{\alpha \in X : b \; or \; c \equiv 1 (mod \; 4)\}$ and $-S = \{\alpha \in X : b \; or \; c \equiv -1 (mod \; 4)\}$. Then $S \cup x(S)$ and $-S \cup x(-S)$ are both disjoint H-subsets of $X \cup x(X)$. Consequently the action of H on $X \cup x(X)$ is intransitive.

Proof: As each $g \in H$ is a word in x, y and y^2. Also we know that $x^{-1} = x$, $y^{-1} = y^3$, $(y^2)^{-1} = y^2$, $(xy)^{-1} = y^3 x$, $(yx)^{-1} = xy^3$ and $(y^2 x)^{-1} = xy^2$. Thus if $\alpha \in S$, then it follows by Table, $y^2(\alpha)$, $xy(\alpha)$ and $yx(\alpha)$ belong to S and hence $y^3 x(\alpha)$ and $xy^3(\alpha) \in S$. However $x(\alpha)$, $y(\alpha)$ and $y^2 x(\alpha)$ does not belong to S and hence $y^3(\alpha)$ and $xy^2(\alpha)$ does not belong to S. Thus by Lemma 2.1 and Table given before Lemma 2.1, $S \cup x(S)$ is an H-subset of $X \cup x(X)$. Similarly, $-S \cup x(-S)$ is an H-subset of $X \cup x(X)$.

If $n = 2^k p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_1, p_2, ..., p_r$ are distinct odd primes and $k = 0 \; or \; 1$ then, by Theorem 2.3, $Q^{**}(\sqrt{n})$ is an H-subset of $Q''(\sqrt{n})$. But if $k \geq 2$, then it is easy to see that $Q^{**}(\sqrt{n})$ is not an H-subset of $Q''(\sqrt{n})$. However, we prove that a superset of $Q^{**}(\sqrt{n})$ is an H-subset of $Q''(\sqrt{n})$. For this, we need to establish the following results!

Lemma 3.6 Let $n = 2^k p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_1, p_2, ..., p_r$ are distinct odd primes and $k = 0 \; or \; 1$. Then

1. $Q^{**}(\sqrt{4n}) = Q''(\sqrt{n}) \setminus Q^*(\sqrt{n})$ and
2. $$Q^*(\sqrt{n}) \setminus Q^{**}(\sqrt{n}) = \{\frac{\alpha}{2} : \alpha = \frac{2a+\sqrt{4n}}{c} \in Q^*(\sqrt{4n}) \setminus Q^{**}(\sqrt{4n})\}$$

Proof: 1. Let $\frac{a+\sqrt{4n}}{c} \in Q^*(\sqrt{4n}) = \{\frac{a+\sqrt{4n}}{c} \in Q^*(\sqrt{4n}) \; and \; 2 | c\}$. Then $\frac{a^2-4n}{c}$ and $\frac{c}{2}$ are both integers and $(a, \frac{a^2-4n}{c}, c) = 1$. As c and $4n$ are both even, so a must be even. Let $a = 2a', c = 2c'$. Then $\frac{a^2-4n}{c} = 2(\frac{a'^2-n}{c'})$ must be odd as otherwise $(a, \frac{a^2-4n}{c}, c) \neq 1$. So $c' = 2c''$. This shows that $\frac{(a')^2-n}{c''}$ is an integer, while $\frac{(a')^2-n}{c'}$ is not an integer for otherwise $\frac{a^2-4n}{c}$ is not odd, a contradiction. Also $(a, \frac{a^2-4n}{c}, c) = 1 \Leftrightarrow (a', \frac{(a')^2-n}{c''}, c'') = 1$. Therefore

$\frac{a+\sqrt{4n}}{c} = \frac{a'+\sqrt{n}}{c'} = \frac{a'+\sqrt{n}}{c''}$ belongs to $Q^*(\sqrt{n})$. Thus $\frac{a+\sqrt{4n}}{c}$ belongs to $Q''(\sqrt{n})\setminus Q^*(\sqrt{n})$.

Conversely let $\frac{a+\sqrt{n}}{2c} \in Q''(\sqrt{n})\setminus Q^*(\sqrt{n})$. Then, by Lemma 2.1, $\frac{a+\sqrt{n}}{2c} \in Q^*(\sqrt{n})$ such that $\frac{a^2-n}{c}$ is odd and hence $\frac{a+\sqrt{n}}{2c} = \frac{2a+\sqrt{4n}}{4c}$ belongs to $Q^*(\sqrt{4n})$. Obviously $\frac{a+\sqrt{n}}{2c}$ belongs to $Q^{**}(\sqrt{4n})$. This completes the first part of Lemma 3.6.

2. We now prove that $\{\frac{\alpha}{2} : \alpha = \frac{2a+\sqrt{4n}}{c} \in Q^*(\sqrt{4n})\setminus Q^{**}(\sqrt{4n})\} = Q^*(\sqrt{n})\setminus Q^{**}(\sqrt{n})$. For this, let $\frac{2a+\sqrt{4n}}{c} \in Q^*(\sqrt{n})\setminus Q^{**}(\sqrt{n})$. Then $\frac{4a^2-4n}{c}$ is an integer and $(2a, \frac{4a^2-4n}{c}, c) = 1 \Leftrightarrow (a, \frac{a^2-n}{c}, c) = 1$. This implies that $\frac{2a+\sqrt{4n}}{2c} = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})\setminus Q^{**}(\sqrt{n})$.

Conversely, suppose that $\frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})\setminus Q^{**}(\sqrt{n})$. Then clearly c is odd and $(a, \frac{a^2-n}{c}, c) = 1$. Also $(a, \frac{a^2-n}{c}, c) = 1 \Leftrightarrow (2a, \frac{4a^2-4n}{c}, c) = 1$. Thus $\frac{a+\sqrt{n}}{c} = \frac{2a+\sqrt{4n}}{2c} = \frac{1}{2}(\frac{2a+\sqrt{4n}}{c})$, where $\frac{2a+\sqrt{4n}}{c} \in Q^*(\sqrt{n})\setminus Q^{**}(\sqrt{4n})$. This completes the proof. W

The following lemma is an extension of Lemma 3.6 for all $n \equiv 0(mod\ 4)$ and its proof is analogous to the proof of above lemma.

Lemma 3.7 Let $n \equiv 0(mod\ 4)$. Then

1.
$$\left(Q^*(\sqrt{\frac{n}{4}})\setminus Q^{**}(\sqrt{\frac{n}{4}})\right) \cup Q^{**}(\sqrt{4n}) = Q''(\sqrt{n})\setminus Q^*(\sqrt{n})$$
and

2.
$$Q^*(\sqrt{n})\setminus Q^{**}(\sqrt{n}) = \{\frac{\alpha}{2} : \alpha = \frac{2a+\sqrt{4n}}{c} \in Q^*(\sqrt{4n})\setminus Q^{**}(\sqrt{4n})\}$$

Theorem3.8 Let $n \equiv 1, 2\ or\ 3(mod\ 4)$. Then $Q^{**}(\sqrt{4n}) \cup \{\frac{\alpha}{2} : \alpha = \frac{2a+\sqrt{4n}}{c} \in Q^*(\sqrt{4n})\setminus Q^{**}(\sqrt{4n})\}$ is an H-subset of $Q''(\sqrt{n})$.

Proof: By Lemma 3.6, $Q^{**}(\sqrt{4n}) = Q''(\sqrt{n})\setminus Q^*(\sqrt{n})$ and $\{\frac{\alpha}{2} : \alpha = \frac{2a+\sqrt{4n}}{c} \in Q^*(\sqrt{4n})\setminus Q^{**}(\sqrt{4n})\} = Q^*(\sqrt{n})\setminus Q^{**}(\sqrt{n})$. Thus $Q''(\sqrt{n})\setminus Q^{**}(\sqrt{n}) = Q^{**}(\sqrt{4n}) \cup \{\frac{\alpha}{2} : \alpha = \frac{2a+\sqrt{4n}}{c} \in Q^*(\sqrt{4n})\setminus Q^{**}(\sqrt{4n})\}$ is an H-subset of $Q''(\sqrt{n})$ if and only if $n \not\equiv 0(mod\ 4)$. Also since $Q^*(\sqrt{n})$ is not H-subset so $Q^*(\sqrt{n})\setminus Q^{**}(\sqrt{n})$ and $Q''(\sqrt{n})\setminus Q^*(\sqrt{n})$ are not H-subsets of $Q''(\sqrt{n})$. By Theorems 2.2, 2.3 we know that $Q''(\sqrt{n})\setminus Q^*(\sqrt{n})$ is an H-subset of $Q''(\sqrt{n})$. Thus $Q^{**}(\sqrt{4n}) \cup \{\frac{\alpha}{2} : \alpha = \frac{2a+\sqrt{4n}}{c} \in Q^*(\sqrt{4n})\setminus Q^{**}(\sqrt{4n})\}$ is an H-subset of $Q''(\sqrt{n})$. W

The following remark is an immediate consequence of Lemma 3.6 and Theorem 3.8.

Remark 3.9 Let $n \not\equiv 0(mod\ 4)$. Then $Q''(\sqrt{n}) = Q^{**}(\sqrt{n}) \cup Q^{*:}(\sqrt{4n})$, where $Q^{*:}(\sqrt{4n}) = \left(Q^*(\sqrt{n})\setminus Q^{**}(\sqrt{n})\right) \cup Q^{**}(\sqrt{4n})$.

The following theorem is an extension of Theorem 3.8 for all $n \equiv 0(mod\ 4)$ whose proof is analogous to the proof of above Theorem.

Theorem 3.10 Let $n \equiv 0(mod\ 4)$. Then $Q^{**}(\sqrt{4n}) \cup \{\frac{\alpha}{2} : \alpha = \frac{2a+\sqrt{4n}}{c} \in Q^{*}(\sqrt{4n}) \setminus Q^{**}(\sqrt{4n})\}$ is an H-subset of $Q''(\sqrt{n})$. W

Theorem 3.11 Let $n \equiv 0(mod\ 4)$ and $\alpha(a,b,c) \in Q^{*}(\sqrt{n})$. Then:

1. If a is odd then $\frac{\alpha}{2}$ belongs to $Q^{**}(\sqrt{4n})$.

2. If a is even then $\frac{\alpha}{2}$ belongs to $Q^{*}(\sqrt{\frac{n}{4}}) \setminus Q^{**}(\sqrt{\frac{n}{4}})$ or $Q^{**}(\sqrt{4n})$ according as $\alpha \in Q^{*}(\sqrt{n}) \setminus Q^{**}(\sqrt{n})$ or $\alpha \in Q^{**}(\sqrt{n})$.

Proof: Let $n \equiv 0(mod\ 4)$. Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^{*}(\sqrt{n})$. Then we have the following

1. If a is odd then $(a^2 - n)$ is odd. So b cannot be even. Therefore, by second part of Lemma 2.1, $\frac{\alpha}{2}$ belongs to $Q^{**}(\sqrt{4n})$.

2. If a is even then $(a^2 - n) \equiv 0(mod\ 4)$. So b, c cannot be both even, as otherwise $(a,b,c) \neq 1$. Thus exactly one of b, c is even. Therefore, again by second part of Lemma 2.1, if b is odd then $\frac{\alpha}{2}$ belongs to $Q^{**}(\sqrt{4n})$. If b is even then, from the proof of Lemma 3.6(2), $\frac{\alpha}{2}$ belongs to $Q^{*}(\sqrt{\frac{n}{4}}) \setminus Q^{**}(\sqrt{\frac{n}{4}})$. That is, $\frac{\alpha}{2}$ belongs to $Q^{*}(\sqrt{\frac{n}{4}}) \setminus Q^{**}(\sqrt{\frac{n}{4}})$ or $Q^{**}(\sqrt{4n})$ according as $\alpha \in Q^{*}(\sqrt{n}) \setminus Q^{**}(\sqrt{n})$ or $\alpha \in Q^{**}(\sqrt{n})$. W

The following example illustrates the above theorem.

Example 3.12 Let $n = 8$. Then $\alpha = \frac{1+\sqrt{8}}{1} \in Q^{*}(\sqrt{8})$ but $\frac{\alpha}{2} = \frac{1+\sqrt{8}}{2} = \frac{2+\sqrt{32}}{4} \in Q^{**}(\sqrt{32})$. Also $\beta = \frac{2+\sqrt{8}}{1} \in Q^{*}(\sqrt{8})$ but $\frac{\beta}{2} = \frac{1+\sqrt{2}}{1} \in Q^{*}(\sqrt{2}) \setminus Q^{**}(\sqrt{2})$. Similarly $\gamma = \frac{2+\sqrt{8}}{4} \in Q^{**}(\sqrt{8})$ whereas $\frac{\gamma}{2} = \frac{4+\sqrt{32}}{16} \in Q^{**}(\sqrt{32})$.

By summarizing the above results we have the following

Theorem 3.13 Let $n \equiv 0(mod\ 4)$. Then $Q''(\sqrt{n}) = Q^{*:}(\sqrt{n}) \cup Q^{*:}(\sqrt{4n})$, where $Q^{*:}(\sqrt{4n}) = \left(Q^{*}(\sqrt{n}) \setminus Q^{**}(\sqrt{n})\right) \cup Q^{**}(\sqrt{4n})$ and $Q^{*:}(\sqrt{n}) = (Q^{*}(\sqrt{\frac{n}{4}}) \setminus Q^{**}(\sqrt{\frac{n}{4}})) \cup Q^{**}(\sqrt{n})$.

Proof: Follows from Lemma 3.7 and Theorem 3.10. W

We conclude this paper with the following examples for illustration of Remark 3.9 and Theorem 3.13. For $n = 2$, $4n = 8$, $Q^{*:}(\sqrt{8}) = (\sqrt{2})^H \cup (-\sqrt{2})^H$, $Q^{*:}(\sqrt{32}) = (\sqrt{8})^H \cup (-\sqrt{8})^H$. So $Q''(\sqrt{8})$ has exactly 4 orbits under the action of H. Also if $n = 3$, $4n = 12$, $Q^{*:}(\sqrt{12}) = (\sqrt{3})^H \cup (-\sqrt{3})^H$, $Q^{*:}(\sqrt{48}) = (\sqrt{12})^H \cup (-\sqrt{12})^H$. So $Q''(\sqrt{12})$ has exactly 4 orbits under the action of H. Similarly if $n = 5$, $4n = 20$, $Q^{*:}(\sqrt{20}) = (\sqrt{5})^H \cup (-\sqrt{5})^H$, $Q^{*:}(\sqrt{80}) = (\sqrt{20})^H \cup (-\sqrt{20})^H$. So $Q''(\sqrt{20})$ has exactly 4 orbits under the action of H.

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