

LOCAL MESHLESS ALGORITHM FOR TOTAL VARIATION-BASED MODEL FOR SIGNAL RESTORATION HAVING ADDITIVE NOISE

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ABSTRACT: A noisy signal converts the signal to another signal containing fluctuation called signal fluctuation. This fluctuation in signal is a common factor in signal processing. In this article, we introduce a Local Meshless Collocation Method (LMCM) for the numerical solution of the Euler-Lagrange Partial Differential Equation (EL-PDE) associated with the Total Variation (TV)-based model for removing additive noise from the given data signals. This method uses the multi-quadric Radial Basis Function (MQ-RBF) as the basis function. These features will eliminate the fluctuations from the given noisy signals well due to the meshless applications used in the suggested method. The experimental result demonstrates that the proposed LMCM will perform well in terms of Signal to Noise Ratio (SNR) compared to another traditional mesh-based scheme for different basis functions. Moreover, the results indicate that the LMCA is faster regarding computation time (CUP time) and requires fewer iterations for convergence than that of conventional method.

Keywords: Euler Lagrange PDE, Additive noise, Signal, PDE, TV- Regularization, SNR, Radial Basis Function Interpolation, MQ-RBF, Local RBF, Global RBF.

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INTRODUCTION

Signal processing entails the conversion or transformation of data in a manner that enables us to perceive aspects that are not observable through direct observation. Signal processing enables engineers and scientists to examine, enhance, and rectify signals, encompassing scientific data, audio streams, pictures, and video. The issue of signal denoising has extensive applications in various domains of engineering and research, including communications, control, and machine learning.

The majority of the literature on signal denoising is focused on the additive noise model, which is described below.

$$w_0 = w + n. \quad (1)$$

Where w_0 is the given noisy signal degraded by additive noise n and w is the original signal. In literature, other methodologies have been employed to address this issue, including wavelet methodologies [1-3], adaptive smoothing [4,5], stochastic methodologies [6,7], and anisotropic diffusion [8,9].

One of the most known techniques for signal denoising is based on the TV-based regularization model [10]. Recently, variational methods have been employed to address difficulties related to TV models, yielding successful restoration outcomes, as demonstrated in previous studies [11–13]. In traditional variational methods, the PDE is derived from the minimization functional that incorporates additive noise. This approach is employed to achieve a smooth solution for signal

denoising. However, the equation of the EL-PDE when connected to a TV model is invariably a non-linear and non-differentiable PDE. Consequently, the traditional numerical methods were unable to effectively solve the EL-PDE for obtaining a smooth numerical solution, resulting in the loss of fine details in the signal denoising process.

Rodin et al. (ROF) [14] proposed the first model for signal restoration with additive noise, which was based on TV. This model produces good outcomes in the elimination of signal noise. However, it also encounters certain undesirable characteristics such as signal loss and delayed processing due to its nonlinearity and non-differentiability [15–17]. This problem to minimize we used the TV-L1 regularization-based model. The TV-L1 have some positive aspects that give us a smooth solution regarding the restoration of signal containing additive noise [18]. However, this model poses challenges due to its lack of linearity and differentiability in the accompanying PDE, which in turn affects the signal. However, there is still room for enhancement. In this study, we employ the mesh-free RBF collocation approach to mitigate these concerns.

In the 1970s, RBF approaches emerged as a solution to the structural constraints imposed by previous numerical methods. RBF can be used with data that is distributed in several locations [19]. The form of the domain may be chosen by the application rather than the numerical method, resulting in this outcome. Over the past decade, RBFs have been recognized as effective methods for interpolating and approximating smooth

functions with many variables using scattered data sets [20]. In recent times, there has been a growing focus on the advancement of mesh-free approaches that utilize RBF for the computational resolution of PDE. The majority of PDEs have focused on steady-state problems with solutions that are smooth. There is currently a rising interest in using RBF methods for time-dependent PDE problems that have solutions that are smooth enough. The RBF techniques have demonstrated higher accuracy compared to classic numerical strategies such as the finite difference method (FDM), finite element method, finite difference volume method (FVM), and pseudo-spectral method [21-25]. Additionally, RBF techniques offer more points of interest. For further information on RBF collocation approaches, readers should consult references [26-28].

The primary benefits of using RBFs for interpolating scattered data in several dimensions are discussed in [29, 30]. Meshless approaches have been demonstrated to effectively address scientific and engineering difficulties in recent decades. The collocation method-based mesh-free method has become dominant and highly efficient. RBF methods are grid-independent and hence fall under the category of mesh-free approaches. They just apply a set of data points without any information regarding the links between them. The object is positively definite under certain conditions [31,32], and it is invariant to rotation and translation. The RBF approximation is a highly effective technique for modelling smooth functions in complex geometries. This method does not rely on a mesh and can achieve spectral accuracy [33].

RBF methods can be implemented both globally and locally. The global method has advantages like adaptive nature and time computation but has issues with ill condition issues which creates issues with the outcomes. However, the local meshless method is far better than the Global meshless method due to its accuracy and efficiency connected with the local structure-property [34], a local meshless method which is the additional application of this method in comparison to the global meshless method. The utilization of local RBF methods for solving stable and time-dependent PDEs was initially mentioned in references [35,36]. Local radial basis function (RBF) approaches have been recently applied in [37,38]. The primary benefit of utilizing the local technique is the reduced requirement for computer storage and flops.

The paper is organized as follows. Section 2 presents the related work. The first subsection includes the TV-L1 model for signal denoising, and the second subsection contains BR interpolation. Section 3 highlights the mesh-based gradient descent numerical scheme for solution EL-PDE connected with the TV-L1 model. This section also contains the proposed meshless scheme LMCM for the numerical solution of EL-PDE

connected with the TV-L1 model. Section 4 describes the Experimental Results Finally, Sect. 5 conclusions of the paper.

Lecturer Review

Total Variation-Based-L1 Model: Chen et al. [39] introduced the first TV regularization-based model for image restoration with additive noise. This approach addresses specific unwanted factors such as signal loss and delayed computation caused by its nonlinearity and lack of differentiability. The TV-L1 approach for model optimization is not strictly convex, and there is not a unique global minimizer according to reference [40]. The TV-L1 model, similar to the ROF model, employs the same regularization term. However, it utilizes the L1-norm in the fidelity term, as seen in the following equation.

$$W = \min_w \{E(w)\} \int_{\Omega} |\nabla w(x, y)| d\Omega + \lambda \int_{\Omega} |w - w_o| d\Omega \quad (2)$$

Here w_o is given noisy image, w is a denoised image and λ is a regularization parameter. The denoise image w is found the following procedure [41]. Finding EL-PDE equation for the above function. Solving EL-PDE equation using gradient descent method and obtaining the solution of PDEs.

$$TV(w) = \int_{\Omega} |\nabla w| dx dy, \quad (3)$$

Where $|\nabla w| = \sqrt{w_x^2 + w_y^2}$

The corresponding time marching EL- PDE Equation of equation (2)

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left(\frac{\partial w_x}{\sqrt{w_x^2 + w_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{\partial w_y}{\sqrt{w_x^2 + w_y^2}} \right) + \lambda \frac{(w - w_o)}{|w - w_o|} \quad (4)$$

For further details, see [42].

Radial Basis Function Interpolation: Now, let's examine the RBF technique [43]. An RBF is a function that takes real values and whose value is entirely determined by the distance from the origin, so that $\Phi(x) = \Phi(\|x\|)$, alternatively on the distance from some other point c , called center, so that $\Phi(x, c) = \Phi(\|x - c\|)$ Any function Φ that satisfied the property $\Phi(x) = \Phi(\|x\|)$ is called the RBF. Some common basis functions that are used in the RBF collocation method are Multi-quadric (MQ), Inverse Multi-quadric (IMQ), Gaussian (GA), Poly-harmonic spline, and thin plate splines (TPS) for further information regarding the basis function. The norm $\| \cdot \|$ is always Euclidean distance. RBF methods in general can be found in recent research monographs on RBF methods [43]. Local RBF is just like the global RBF method for given data points or evaluation points $\{x_i, Z_i\}_{i=1}^N$. The only difference is that in the global RBF method, we are free to select Centre points. But in the local RBF method, we are not free to select the Centre points. For this, we need a stencil value

n and a point from evaluation points through which the Centre values are measured [44].

But for $\{x_{i,j}, Z_{i,j}\}_{j=1}^M$ Centre points and $\{x_i^c, Z_i\}_{i=1}^M$ evaluation points.

$$s(x) = \sum_{j=1}^n \rho_j \phi(\|x - x_j^c\|_2), \quad (5)$$

Which result in $A = [\Phi_{ij}] = (A_{ij}) \in R^{M \times M}$

$$A = \begin{bmatrix} \Phi(\|x_1^c - x_1^c\|_2), \Phi(\|x_1^c - x_2^c\|_2) \cdots \Phi(\|x_1^c - x_M^c\|_2) \\ \Phi(\|x_2^c - x_1^c\|_2), \Phi(\|x_2^c - x_2^c\|_2) \cdots \Phi(\|x_2^c - x_M^c\|_2) \\ \vdots \\ \Phi(\|x_M^c - x_1^c\|_2), \Phi(\|x_M^c - x_2^c\|_2) \cdots \Phi(\|x_M^c - x_M^c\|_2) \end{bmatrix} \quad (6)$$

And

$$\rho = (\rho_1, \rho_2, \dots, \rho_n)^t, w_o = (w_o(x_1), w_o(x_2), \dots, w_o(x_n))^t, \text{ are the } M \times 1 \text{ matrices.}$$

$$\begin{bmatrix} \Phi(\|x_1^c - x_1^c\|_2), \Phi(\|x_1^c - x_2^c\|_2) \cdots \Phi(\|x_1^c - x_M^c\|_2) \\ \Phi(\|x_2^c - x_1^c\|_2), \Phi(\|x_2^c - x_2^c\|_2) \cdots \Phi(\|x_2^c - x_M^c\|_2) \\ \vdots \\ \Phi(\|x_M^c - x_1^c\|_2), \Phi(\|x_M^c - x_2^c\|_2) \cdots \Phi(\|x_M^c - x_M^c\|_2) \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_N \end{bmatrix} = \begin{bmatrix} w_o(x_1) \\ w_o(x_2) \\ \vdots \\ w_o(x_N) \end{bmatrix}, \quad (7)$$

This can be written in equation form as under,

$$A\rho = w_o \quad (8)$$

Where A is called $M \times M$ system matrix. So, the equation (8) can be written as under.

$$\rho = A^{-1}w_o \quad (9)$$

The invertible matrix A (non-singular matrix) is depends upon the basis function used in RBF in A .

Now for $\{x_j^c, w_j\}_{j=1}^M$ stencil points and $\{x_i, w_j\}_{i=1}^N$ evaluation points. The interpolation condition is for $s(x) = \sum_{i=1}^N \rho_i \phi(\|x_i - x_j^c\|_2) = w_{oi} \quad i = 1, 2, 3, \dots, N,$

$$j = 1, 2, 3, \dots, M \text{ and } M < N. \quad (10)$$

So (10) results in the following system of equations.

$$H\rho = w_o \quad (11)$$

Here $H = [h_{i,j}] = [\phi(\|x_i - x_j^c\|_2)]$ for $i = 1, 2, 3, \dots, N$ and $j = 1, 2, 3, \dots, M$

$$H = \begin{bmatrix} \Phi(\|x_1 - x_1^c\|_2), \Phi(\|x_1 - x_2^c\|_2) \cdots \Phi(\|x_1 - x_M^c\|_2) \\ \Phi(\|x_2 - x_1^c\|_2), \Phi(\|x_2 - x_2^c\|_2) \cdots \Phi(\|x_2 - x_M^c\|_2) \\ \vdots \\ \Phi(\|x_N - x_1^c\|_2), \Phi(\|x_N - x_2^c\|_2) \cdots \Phi(\|x_N - x_M^c\|_2) \end{bmatrix} \quad (12)$$

And H is $N \times M$ matrix is called an evaluation matrix. also

$$\rho = (\rho_1, \rho_2, \rho_3, \dots, \rho_M)^t \text{ and}$$

$$w_o = (w_o(x_1), w_o(x_2), w_o(x_3), \dots, w_o(x_n))^t$$

So $w_o = H\rho$ (13)

put the value of ρ from equation (9) in equation (13)

$$w_{o\text{approx}} = HA^{-1}w_o \quad (14)$$

$$\text{Let } HA^{-1} = D \quad (15)$$

$$w_{o\text{approx}} = Dw_o \quad (16)$$

Equation (16) $N \times 1$ system and is called the approximation solution of the function w_{oi} and given point $x_i \quad i = 1, 2, 3, \dots, N$.

Numerical Methods Used To Solve TV-L1 Model Signal Restoration

Traditional Mesh-Based Method M1: The TV-L1 model is employed for the restoration of signals that have additive noise, and it has demonstrated favorable recovery outcomes on 1-D signals. So, the TV-L1 model can be mathematically written as,

$$\min_w \{E(w)\} = \int_{\Omega} |\nabla w(x, y)| d\Omega + \int_{\Omega} |w - w_o| d\Omega. \quad (17)$$

The EL-PDE equation (17) of the TV-L1 model is below.

$$\lambda \frac{(w - w_o)}{|w - w_o|} - \frac{w_{xx}w_y^2 - 2w_xw_yw_{xy} + w_{yy}w_x^2}{(w_x^2 + w_y^2)^{\frac{3}{2}}} = 0. \quad (18)$$

Solving the EL- PDE equation (18) using the gradient-descent method, we have

$$\frac{\partial w}{\partial t} = - \frac{\partial E}{\partial w} = \frac{w_{xx}w_y^2 - 2w_xw_yw_{xy} + w_{yy}w_x^2}{(w_x^2 + w_y^2)^{\frac{3}{2}}} - \lambda \frac{(w - w_o)}{|w - w_o|}. \quad (19)$$

By employing Euler's Forward Method, the equation (19) mentioned above is reformulated.

$Num =$

$$\left((w_{xx})_{i,j}^{(n)} ((w_y)_{i,j}^{(n)})^2 - 2(w_x)_{i,j}^{(n)} (w_y)_{i,j}^{(n)} (w_{xy})_{i,j}^{(n)} + (w_{yy})_{i,j}^{(n)} ((w_x)_{i,j}^{(n)})^2 \right) \quad (20)$$

$$Den = \left(((w_x)_{i,j}^{(n)})^2 + ((w_y)_{i,j}^{(n)})^2 \right)^{\frac{3}{2}} + 0.01 \quad (21)$$

$$w_{i,j}^n = w_{i,j}^n + \Delta t \left(\frac{Num}{Den} - \lambda \frac{(w_{i,j}^{(n)} - w_o)}{|w_{i,j}^{(n)} - w_o|} \right) \quad (22)$$

For more detail see [45].

Proposed Local Meshless Collocation Method (LMCM) M2:

In this subsection, we present the Local meshless approach, which utilizes the RBF collocation scheme to numerically solve the TV-L1 model equation (4) and obtain the resultant signal from the provided noisy image specified in model equation (2).

Suppose $\{x_i\}_{i=1}^N$ is the N distinct evaluation points in $\Omega \subseteq R^2$ So, for any RBF the following equations are satisfied, $\phi(w) = \|w\|_2$ in R^2 , i.e. $w = (x, y)$ for $\{x_i\}_{i=1}^N$, given N_c centers, given as

$$S(x) = \sum_{i=1}^{N_c} \rho_i \phi(\|x - x_i\|_2) \quad (23)$$

The ρ_i coefficients in the RBF are defined by implementing the interpolation condition $s(x_{i=1}) = w_o$ on a collection of points, which typically coincide with

N_c centres. The interpolation condition at centers results in a linear system of size $N_c \times N_c$.

$$C\rho = w_o \quad (24)$$

The expansion coefficients of ρ need to be determined. Where $\rho = (\rho_1, \rho_2, \dots, \rho_{N_c})^t$ and $w_o = (w_o(x_1), w_o(x_2), \dots, w_o(x_n))^t$ are $N_c \times 1$ matrices. The matrix C is referred to as the interpolation matrix or system matrix, and it is defined as follows:

$$C = [\phi_{i,j}] = [\phi(\|x_i - xc_j\|_2)] \quad 1 \leq i, j \leq N_c.$$

The system matrix C is a square matrix with dimensions $N_c \times N_c$ and it is invertible [31] due to its positive definite nature [20].

$$\rho = C^{-1}w_o \quad (25)$$

where ρ is $N_c \times 1$ matrix. The interpolant is evaluated using equation (23) at N evaluation points $(\{x_i\}_{i=1}^N)$, through forming $(N \times N_c)$ evaluation matrix D which is given as

$$D = (\phi_{i,j}) = [\phi(\|x_i - xc_j\|_2)] \quad \text{for } i = 1, 2, \dots, N, \quad j = 1, 2, \dots, N_c$$

The interpolant is subsequently assessed at the N point using matrix-vector multiplication to generate w as described.

$$w = D\rho \quad (26)$$

Now, from Equation. (25) and equation (26) we have,

$$w = DC^{-1}w_o$$

Or

$$w = Hw_o \quad \text{where } H = DC^{-1} \quad (27)$$

It provides an approximate solution at any given point inside the domain Ω . Where is the $N \times 1$ matrix located.

As in the TV-L1 model, the time-depending equation (4),

$$\frac{dw^{(n+1)} - w^{(n)}}{dt} = \frac{(w_{xx}^{(n)} + w_{yy}^{(n)})((w_x^{(n)})^2 + (w_y^{(n)})^2) - (2w_x^{(n)}w_y^{(n)}(w_x^{(n)}w_y + w_xw_y^{(n)}) + w_x^2w_{xx}^{(n)} + w_y^2w_{yy}^{(n)})}{((w_x^{(n)})^2 + (w_y^{(n)})^2)^{\frac{3}{2}}} \quad \text{compute } w^{(n+1)} \text{ according to equation (25) by Kansa method.}$$

$$\lambda \frac{(w^{(n)} - w_o)}{|w^{(n)} - w_o|} \quad t > 0, \quad (x, y) \in R, \quad (28) \quad \text{The}$$

current non-linear system of equations exhibits the restoration PDE.

$$\begin{aligned} L(w^{(n)})w^{(n+1)} &= L(w^{(n)})w^{(n)} \\ &+ dt \left[(w_{xx}^{(n)} + w_{yy}^{(n)}) \left((w_x^{(n)})^2 + (w_y^{(n)})^2 \right) \right. \\ &- (2w_x^{(n)}w_y^{(n)}(w_x^{(n)}w_y + w_xw_y^{(n)}) \\ &\left. + w_x^2w_{xx}^{(n)} + w_y^2w_{yy}^{(n)}) \right] - \\ dt \left[L(w^{(n)})\lambda \frac{(w^{(n)} - w_o^{(0)})}{|w^{(n)} - w_o^{(0)}|} \right] \end{aligned} \quad (29)$$

$$t > 0, \quad (x, y) \in R, \quad \text{where } L(w) = (w_x^2 + w_y^2)^{\frac{3}{2}}, w_x = Hw_o, w_{xx} =$$

$$H_{xx}w_o, w_y = H_yw_o, w_{yy} = H_{yy}w_o, \frac{\partial w}{\partial n} = w_n = H_nw_o \text{ and } w^{(0)} = w_o$$

As the RBF used in the Collocation approach may not always meet the handling Equation (29), we have greater flexibility in selecting an RBF. The MQ-RBF is the most widely used Kansa technique. It is known for its spectral accuracy, provided that an adequate shape parameter is selected. In our approach, M2, the shape parameter utilized in RBF is crucial for achieving smoothness. Our suggested methodology achieves superior accuracy and smoothness in signal denoising and requires fewer iterations for convergence, resulting in improved CPU time efficiency. The shape parameter and fitting parameter λ in this technique are contingent upon the signal's size and the noise level present in the signal.

Algorithm 1: Algorithm LMCA M2

RBF

1. Select $N = N_c, n$ number of pixel points where N and N_c are called the total number of pixel and center pixel points which are used in the RBF approximation process with stencil p with a number of points n_1 .
2. Find the ρ according to equation (25) by MQ-RBF by using step (1).
3. Find w according to equation (27) by MQ-RBF by using steps (1) and (2).

TV filtering

4. Initialize the values of $\lambda, \varepsilon, c, dt$ and w_o .
5. Set $nasN_c$ centers pixel points i.e. $x_{c1} \leq x_{c2} \leq x_{cn}, \dots, x_{cn}$, set $n = 0$.
6. At $n = 0$, put w as MQ-RBF matrix in equation (29) from equation (27). Where we choose $w^{(0)} = w_o$.
7. For each center point x_{ci} , for $i = 1, 2, 3, \dots, n$, compute $w^{(n+1)}$ according to equation (25) by Kansa method.
8. $\frac{\|w^{(n+1)} - w^{(n)}\|}{\|w^{(n)}\|} \leq \varepsilon = 10^{-5}$ go to step (10).
9. Go to step (7).
10. End.
11. Output $w = w^{(n+1)}$.

Experiential Analysis: This section presents some experimental results and analyses that are carried out on various signals like real signals to verify the performance of the suggested meshless algorithm M2. There is some noisy signals are used to evaluate the performance of M1 and M2. These test signals here are displayed in the figure. In this study, we compare noisy signals in a test containing salt and pepper noise ($P = 0$ variation P). To verify the signal restoration outcomes of the proposed LMCM M2 and compare its results with the traditional mesh-based technique M1. We select the signal size with $N = N_c$ size and use the suggested meshless techniques M2. The SNR is taken into consideration to quantify the denoised signal. This measure is widely used for the

quality of the restored signal.

SNR is defined as

$$SNR = 10 \times \log_{10} \frac{\|w - w_o\|}{\|n - n_o\|} \quad (30)$$

Where w_o is the final denoised signal, ($M \times N$ denotes the signal data size taken from the image), w is the original signal that was provided. The greater the SNR value better the restoration performance.

$$\frac{\|w^{(n+1)} - w^{(n)}\|}{\|w^{(n)}\|} \leq \varepsilon, \quad (31)$$

where ε is the highest acceptable level of inaccuracy. Here, it is set to be 10^{-1} . In this study, we employ the MQ-RBF method to evaluate and contrast the outcomes of the M2 and M1 schemes. The MQ-RBF is defined for each point (u_i, v_j) , as follows:

$$\begin{aligned} \phi_j(u, v) &= \sqrt{c^2 + w_j^2} \\ &= \sqrt{c^2 + ((u - u_j)^2 + (v - v_j)^2)}, \end{aligned}$$

Inverse Multi-quadric (IMQ-RBF) is given as under.

$$\phi_j(u, v) = \frac{1}{\sqrt{c^2 + w_j^2}} = \frac{1}{\sqrt{c^2 + (u - u_j)^2 + (v - v_j)^2}}$$

Also, the Gaussian (GA)-RBF is given as under.

$$\phi_j(u, v) = e^{-c^2 w_j^2} = e^{-c^2 \{(u - u_j)^2 + (v - v_j)^2\}},$$

where

$$w_j = \sqrt{(u - u_j)^2 + (v - v_j)^2},$$

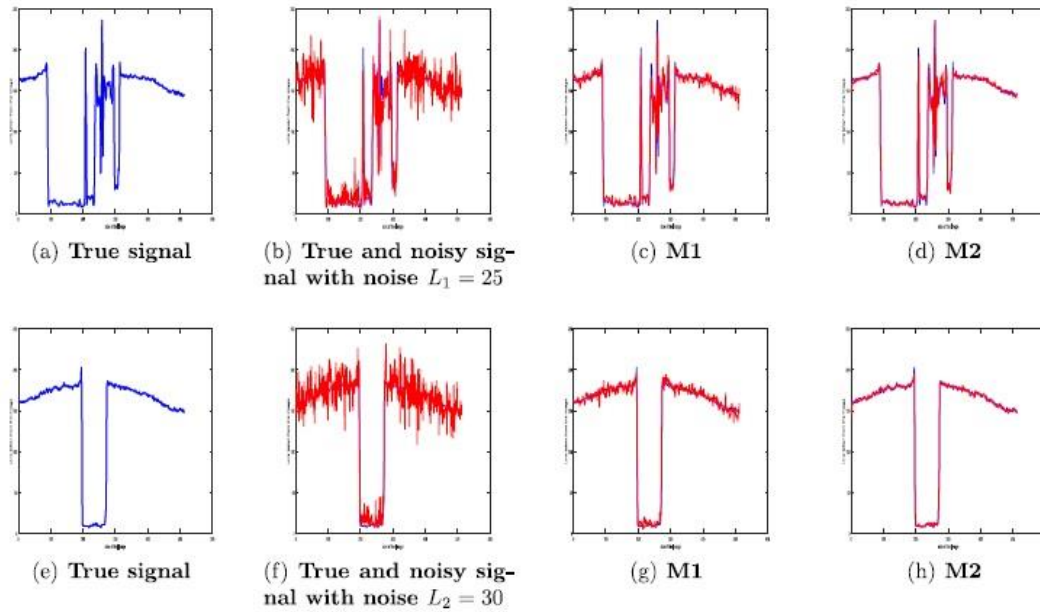


Fig 1: Left to right, signals (a) and (e) show 180th and 100th the original signal taken from the Cameraman image, (b) and (f) represent the original and noisy signals having additive Gaussian noise $L1 = 25\%$ and salt and pepper noise $L2 = 30\%$, respectively; (c) and (g) indicate the original and denoised signals using M1; (d) and (h) represents the denoised signals using LMCA M2. Here the blue line indicates the original signal while the red line indicates the noisy or restored signal.

Tab 1: SNR values comparison between M1 and M2.

S. No	Signal selected from the Image	Size	M1	M2
			SNR	SNR
1	180 th	512	22.26	22.8
2	100 th	512	23.88	24.47

Tab 2: M1 and M2 comparison regarding the number of iterations and time in seconds required for convergence.

S.No	Signal taken from the Image	Size	M1		M2	
			Iteration	Time(s)	Iteration	Time(s)
1	180 th	512	16	5.05	9	3.06

2	100 th	512	19	7.03	10	4.05
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Test 1: This test indicates the comparison of two mesh-based schemes M1 and meshless scheme M2 for signal restoration having additive Gaussian noise and salt and pepper noise with noise levels $L1 = 25\%$ and $L2 = 30\%$, respectively which are shown along with the original signal in Figures 1(a), 1(e), 1(b), and 1(f), respectively. M1 restored the signal but due to the nonlinearity and non-differentiable of the associated PDE connected with the additive noise removal model by mesh-based method cannot provide good restoration results even though TV-regularization can restore the image well. These results are shown in Figures 1(c) and 1(g). Meshless scheme M2 provides better signal restoration results than M1 due to the meshless

applications, localization and adaptive natures, and MQ-RBF connected with LMCM M2. These restoration results are shown in Figures 1(d) and 1(h). Table 1 shows that SNR values of M2 are greater than M1 which indicates the best signal restoration performance of M1 over M1. Also, the number of iterations and CPU time required for convergence of M2 is smaller than M1 which indicates the quick restoration performance of M2 over M1 due to its computational effective property. In this test the best values of $\lambda=12$ and $c=1.16$.

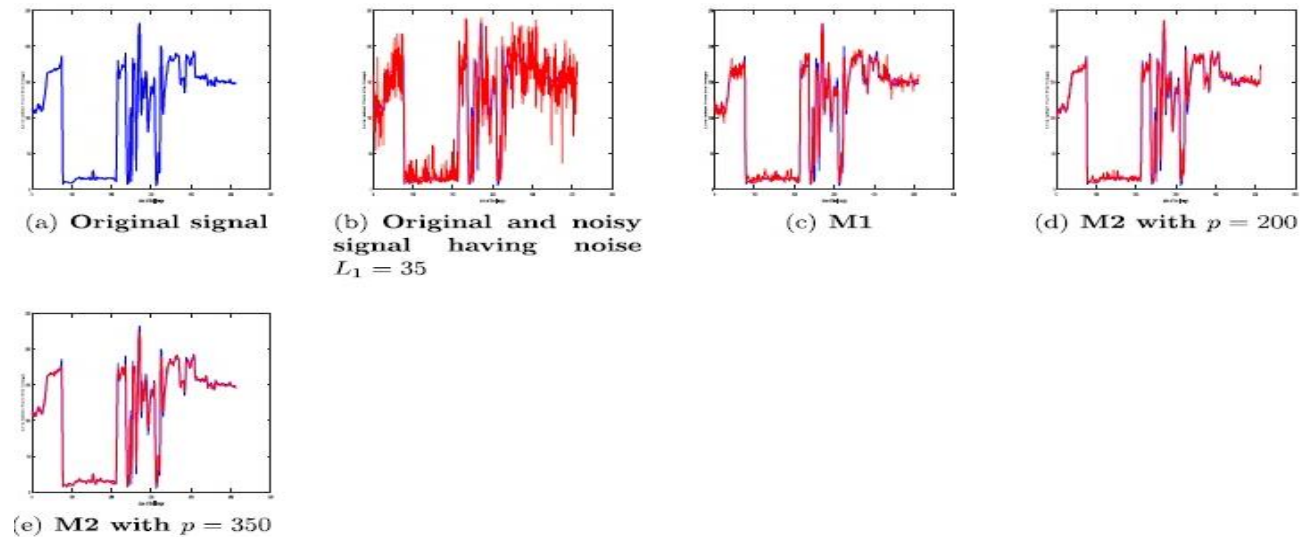


Fig 2: Left to right, (a) shows the 300th original signal taken from Lena image; (b) indicates the original and noisy signal connected additive Gaussian noise $L1 = 35\%$; (c) shows the original and restored signals by M1; while (d) and (e) represent the original and restored signals by M2 connected with centers $p = 200$ and $p = 350$, respectively. Here the blue line indicates the original signal while the red line indicates the noisy or restored signal.

Tab 3: Representation of SNR values of M1 and M2 for image restoration.

S. No	Signal chosen of the Image	Size	M1	M2	
			SNR	SNR(p=200)	SNR(p=350)
1	300 th	512	24.35	24.81	24.98

Tab 4: Comparison of M1 and M2 regarding number of iterations and CPU times in seconds.

S.No	Signal taken from the Image	Size	M1		M2			
			Iteration	time (s)	P=200		p=350	
					Iteration	Time(s)	Iteration	Time(s)
1	300 th	512	17	16.9	11	10.42	7	7.01

Test 2: M1 and M2 for tested in this test for signal restoration with original signal and noisy signal having salt and pepper noise $L = 35\%$. These signals are shown in Figures 2(a) and 2(b). Again, the poor signal restoration is obtained by M2 because of the mesh-based solution of nonlinear and non-differentiable PDE connected with the model. This signal is shown in Figure 2(c). The proposed meshless method M2 is tested for the same shape parameter $c=1.22$ but for two different center points i.e., $p = 200$ and $p = 350$. It can be seen that the signal restoration for selecting more center points produces better restoration performance than selecting smaller center points. But in both cases, the signal restoration results by M2 are better than M1 due to its all meshless application mentioned in test 1. These results are presented in Figures 2(d) and 2(e). Table 3 shows that the SNR values of M2 are greater than the M1 value which shows the good signal restoration performance by

M2 over M1. Table 3 also indicates that the SNR values are greater for center values $p = 350$ than center values $p = 200$ for M2 for the same shape parameter which shows better restoration performance with a greater number of center points than a smaller number of center points. In addition, Table 4 shows that the number of iterations and CPU time required for convergence of M2 is less than M1 which represents the faster image restoration of M2 over M1 due to the meshless applications discussed in test 1. Also, the number of iterations and CUP time of M2 for higher values of centers $p = 350$ are less than the lower values of centers $p = 200$ for the same shape parameter which shows the quick signal restoration performance of M2 for higher values of centers than lower values of centers. The optimal values selected in this test are $\lambda=16$ and $\lambda=1.18$.

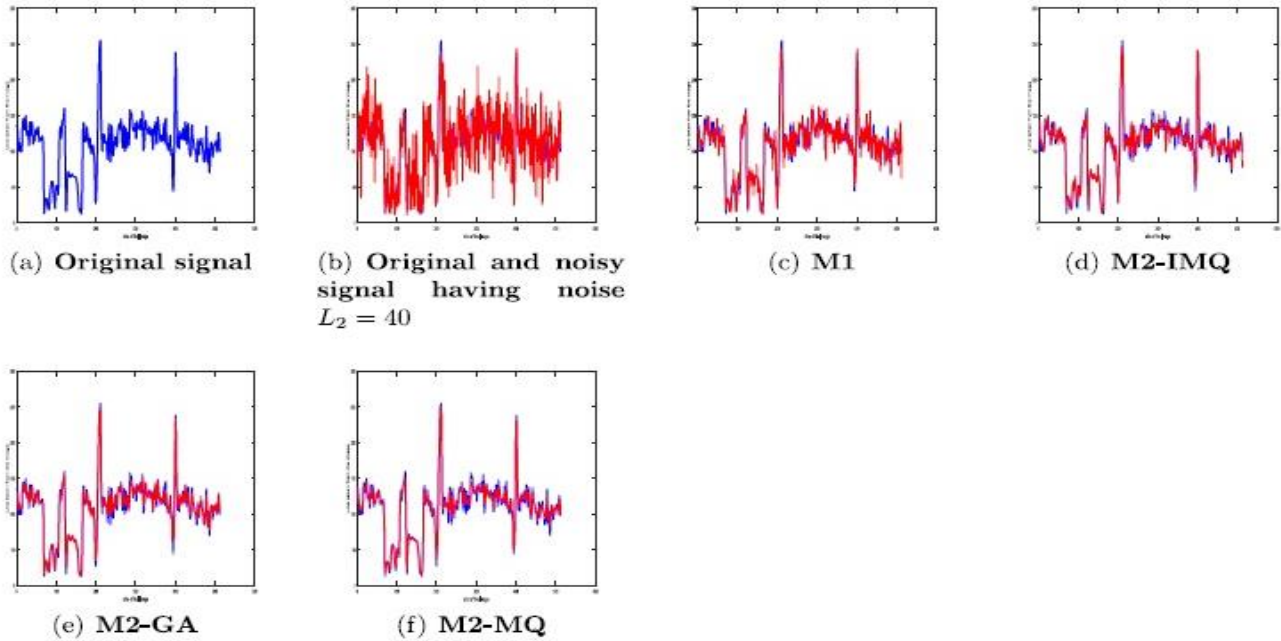


Fig 3: Left to right; Comparison of original and noisy signals taken from Lena image connected with salt and pepper noise $L_2 = 40\%$, these are denoted by (a) and (b); original and restored signals by using M1 and M2 for IMQ-RBF, GA-RBF, and MQ-RBF, respectively. These signals are shown in (c), (d), (e), and (f), respectively. Here the blue line indicates the original signal whereas the red line indicates the noisy or restored signal.

Tab 5: Comparison of SNR values of M1 and M2 which is connected with IMQ-RBF, GA-RBF, and MQ-RBF, respectively.

S. No	Signal selected of the Image	Size	M1	M2(IMQ)	M2(GA)	M2(MQ)
			SNR	SNR	SNR	SNR
1	499	512	23.09	23.76	23.84	24.09

Test 3: In this rest, M2 is compared with M1 for signal restoration from noisy signals having salt and pepper noise with noisy level $L2 = 40\%$. The true and true and noisy signals are shown in Figures 3(a) and 3(b). Again, M2 obtained a better and quicker restoration performance than M1 due to the meshless application used in M2 over M1. These signals are shown in Figures 3(c), 3(d), 3(e), 3(f), and Table 5, respectively. Furthermore, the proposed meshless method M2 is tested on three different basis functions i.e., IMQ-RBF, GA-RBF, and MQ-RBF for signal restoration. It is founded on the experimental results that the signal restoration results obtained by MQ-RBF than the other two basic functions IMQ-RBF and GS-RBF which is the fundamental claim in [30]. Also, the results generated by GA-RBF and better than IMQ-RBF for M2. All these results are shown in Figures 3(d), 3(e), 3(f), and Table 5, respectively. The best selected values of involved parameters are $\lambda=21$ and $c=1.19$.

The main limitation in our work is the selection of optimal values of parameters involved in the proposed LMCM hard to select by “head and trail” rules and also time consuming.

Conclusion: In this article has introduce the Local Meshless Collocation Algorithm (LMCA) for the numerical solution of EL-PDE connected total variation base method and to get smooth solution from regarding signal denoising with additive noise. The LMCA has the ad-vantage for generating quick and bitter restoration performance (signal denoising) due to meshless nature, adaptive nature, MQ-RBF used as basis function, and computational efficiency nature. The associated meshless scheme is checked on various signal and compared their results with traditional method. The experimental results have reviled that the proposed meshless method (LMCM) is far bitter in signal restoration i.e., signal denoising (SNR values) and CPU time compared with some traditional method.

However, the optimal values of parameters involved in the proposed scheme M2 are hard to select. This limitation is under consideration and will be addressed in the subsequent paper.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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Appendix: The derivatives of our M2 for equation (29) are given as under so equation (25)

$$\rho = C^{-1}w_o, \quad (32)$$

When we calculate the derivative at N evaluation points $(\{x_i\}_{i=1}^N)$ and N_c center points $(\{x_j\}_{j=1}^{N_c})$, the result is RBF interpolation.

$$w = \sum_{j=1}^N \rho_j \phi(\|x - x_{cj}\|_2), \quad (33)$$

Or

$$w = D\rho \quad (34)$$

Using the evaluation matrix D with dimensions $(N \times N_c)$. i.e,

$$D = [\phi_{ij}] = \left[\phi(\|x_i - x_{cj}\|_2) \right] \text{ for } i = 1, 2, \dots, N, j = 1, 2, 3, \dots, N_c$$

Then the derivative becomes from equation (33) as follows:

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^N \rho_j \frac{\partial}{\partial x_i} \phi(\|x_i - x_{cj}\|_2), \quad (35)$$

Or

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} D\rho. \quad (36)$$

Where

$$\frac{\partial D}{\partial x_i} = \frac{\partial[\phi_{ij}]}{\partial x_i} = \frac{\partial}{\partial x_i} \left[\phi(\|x_i - x_{cj}\|_2) \right] \text{ for } i = 1, 2, 3, \dots, N, j = 1, 2, 3, \dots, N_c$$

Merging equation (32) and equation (36) we have

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} DC^{-1}w_o. \quad (37)$$

The $\rho = C^{-1}w_o$ equation (37) which is given as under:

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} Hw_o = H_{xi}w_o. \quad (38)$$

Define the differentiable matrix as

$$H_{xi} = \frac{\partial}{\partial x_i} DC^{-1} = D_{xi}C^{-1}. \quad (39)$$

For the second derivative, we have

$$H_{x_ix_i} = \frac{\partial^2}{\partial x_i^2} DC^{-1} = D_{xixi}C^{-1} \quad (40)$$

also

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2}{\partial x_i^2} Hw_o = H_{xixi}w_o \quad (41)$$

The differentiation matrix is valid and unambiguous due to the known fact that the system matrix C is invertible. The chain rule can be used for any RBF, $\phi[r(x)]$, that is sufficiently differentiable.

$$\frac{\partial \phi}{\partial x_i} = \frac{d\phi}{dr} \frac{dr}{dx_i} \quad (42)$$

The first derivative

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}. \quad (43)$$

The second derivative is follows as

$$\frac{\partial^2 \phi}{\partial x_i^2} = \frac{d\phi}{dr} \frac{\partial^2 r}{\partial x_i^2} + \frac{d^2 \phi}{dr^2} \left(\frac{\partial r}{\partial x_i} \right)^2, \quad (44)$$

With

$$\frac{\partial^2 r}{\partial x_i^2} = \frac{1 - \left[\frac{\partial r}{\partial x_i} \right]^2}{r}. \quad (45)$$

Regarding the MQ specifically,

$$\frac{d\phi}{dr} = \frac{d[c^2 + r^2]^{\frac{1}{2}}}{dr} \quad (46)$$

And

$$\frac{d^2 \phi}{dr^2} = \frac{c^2}{[c^2 + r^2]^{\frac{3}{2}}}. \quad (47).$$