DOUBLE WEIBULL DISTRIBUTION: PROPERTIES AND IT'S APPLICATION

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ABSTRACT: Basis on theoretical properties to find out the basic assumptions about double Weibull distribution were determined with different shape and scale parameters were used to find the new distribution with modified approaches of different parameters. Some characteristics of the newly proposed distribution were obtained using the existing distribution. Different properties were used to find out the cumulative distribution function, probability density function (PDF), hazard function, Reverse Hazard and survival functions. The measure of skewness, kurtosis for selected coefficients and parameters of the new distribution were also calculated. To estimate the parameters, the maximum likelihood (ML) technique was used. Mathematica software was used to draw graphical representations for different shape and scale parameters. of statistical application of the results of life data analysis. On the basis of numerical results, we found that the suggested Double Weibull Distribution was more suitable than the existing Weibull probability distributions. in this study.

Keywords: Weibull distribution, Double Weibull Distribution, Maximum Likelihood, Skewness, Moments and Methematica.

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INTRODUCTION

Many continuous distributions have a vast range of application; Most of the distributions are underlying with specific parameters including scale, shape and location parameters. Weibull Distribution is important and well known distributions that attract statisticians, working in various fields of applied statistics as well as theory and methods in modern statistics (Johnson *et al.,* 1994). Different generalizations of the Weibull distribution are available in the literature as Merovci and Elbatal (2015) developed the Weibull-Rayleigh distribution and demonstrated its application using lifetime data. Almalki and Yuan (2013) presented the new modified Weibull distribution by combining the Weibull and the modified Weibull distribution in a serial system. The distribution has a number of special features and ability to fit the data related to various fields like including testing, biology, ecology, economics, hydrology, engineering and business administration. The Weibull distribution is considered as the limit distribution of the smallest or greater value in a sample with sample size ∞. This distribution comprises the Rayleigh and the exponential distributions as superior cases. In 1939 Swedish physicist Waloddi Weibull introduced a distribution named as the Weibull distribution reported by (Balakrishnan and Kocherlakot, 1985) introduced double weibull distribution with the context of order statistics and its estimation like Mean, variances and covariance's of the order statistics tabled for estimations of n. Later on, wavelet shrinkage with double Weibull prior introduced by (Remenyi and Vidakovic, 2015). They proposed two estimators i.e. one based on "double Weibull wavelet Shrinker" posterior mean and the other based on "double Weibull wavelet Shrinker" i.e. larger posterior mode (LPM) and shown calculation. Estimation procedures are carried out for the shifted Weibull distribution, when all its parameters are unidentified (Richard *et al.,* 1994). A model named modified Weibull extension with three parameters is used for growing, bathtub-shaped, or declining failure rate function and the resulting Weibull probability plot is conceived by (Xie *et al.,* 2002).

MATERIALS AND METHODS

The aim of this section was to introduce a new double Weibull distribution with shape (β) and scale parameter (c). The double Weibull distribution was suggested as under.

$$g(x;\beta,c) = c_0 x^{\beta-1} e^{-x^\beta}(1-e^{-x^\beta c^\beta}),\ \beta>0,\ c>0 \quad (1)$$

Where $c_0 = \frac{\beta(1+c^\beta)}{c^\beta}$

Several properties of the new suggested distribution were derived by using different functions; one of the important functions is distribution function used to check the behavior of the distribution, which is as under.

$$F(x;\beta) = 1 - \frac{(1+c^{\beta})e^{-x^{\beta},}}{c^{\beta}} + \frac{e^{-x^{\beta}(1+c^{\beta})}}{c^{\beta}}, \ \beta>0, c>0 \quad (2)$$

The survival function of the proposed distribution was given by

$$S(x) = \frac{e^{-x^{\beta}}}{c^{\beta}}\left[\left(1+c^{\beta}\right) - e^{-x^{\beta}c^{\beta}},\right], \beta>0, c>0 \quad (3)$$

The hazard rate of Double Weibull Distribution (DWD)

$$h(x) = \frac{\beta(1+c^{\beta})x^{\beta-1}[1-e^{-x^{\beta}c^{\beta}}]}{(1+c^{\beta}) - e^{-x^{\beta}c^{\beta}}}, \beta>0, c>0 \quad (4)$$

The reverse Hazard rate function was determined as the relation among the probability density function and it's confirming distribution function. The reverse hazard rate function of DWD

$$R(x) = \frac{\beta(1+c^{\beta})x^{\beta-1}e^{-x^{\beta}}[1-e^{-x^{\beta}c^{\beta}}]}{c^{\beta} - (1+c^{\beta})e^{-x^{\beta}} + e^{-x^{\beta}(1+c^{\beta})}}, \beta>0, c>0 \quad (5)$$

The Mills Ratio

$$m(x) = \frac{(1+c^{\beta}) - e^{-x^{\beta}c^{\beta}}}{\beta(1+c^{\beta})x^{\beta-1}\left[1-e^{-x^{\beta}c^{\beta}}\right]}, \beta>0, c>0 \quad (6)$$

The moment generating function defined as

$$M_X(t) = \sum_{i=0}^{\infty}\frac{(t)^i}{i!\,c^{\beta}}\left[(1+c^{\beta})\Gamma\left(1+\frac{i}{\beta}\right) - \frac{\Gamma(1+i)}{(1+c^{\beta})^{i}}\right] \quad (7)$$

The information generating function derived using Eq. (1)

$$T(s) = \frac{\beta^{s-1}(1+c^{\beta})^{s}}{c^{s\beta}}\sum_{i=0}^{s}(s)_i(-1)^i\,\Gamma\left(s-\frac{s}{\beta}+\frac{2}{\beta}-1\right) \quad (8)$$

Note that the limit of the density function given in (1)

$$x\to 0, \quad g(x;\beta,c) = x\to 0\,\frac{\beta(1+c^{\beta})}{c^{\beta}}\ x^{\beta-1}e^{-x^{\beta}}\left(1-e^{-x^{\beta}c^{\beta}}\right)=0 \quad (9)$$

$$x\to \infty, g(x;\beta,c) = \frac{\beta(1+c^{\beta})}{c^{\beta}} x\to\infty\ x^{\beta-1}e^{-x^{\beta}}\left(1-e^{-x^{\beta}c^{\beta}}\right) = 0 \quad (10)$$

Since $x\to\infty$, $e^{-x^{\beta}} = 0$ and $x\to\infty$, $(1-e^{-x^{\beta}c^{\beta}}) = 1$.

Taken ln of Eq. (1) on both sides

$$ln(x;\beta,c) = ln\left(\frac{\beta(1+c^{\beta})}{c^{\beta}}\right) + ln\left(x^{\beta-1}\right) - x^{\beta} + ln\left(1-e^{-x^{\beta}c^{\beta}}\right) \quad (11)$$

Differentiating Eq. (11) with respect to x

$$\frac{\partial}{\partial x} ln\, g(x;\beta,)\, c) = \frac{\beta-1}{x} - \beta x^{\beta-1} + \frac{e^{-x^{\beta}c^{\beta}}\beta x^{\beta-1}c^{\beta}}{1-e^{-x^{\beta}c^{\beta}}} \quad (12)$$

Descriptive statistics like mean, standard deviation and moment ratios of the double Weibull distribution were obtained by using different shape and scale parameters for validation (Table 3 and 4).

The r^th moment of DWD

$$\mu_r^{/} = \frac{\Gamma\left(1+\frac{r}{\beta}\right)}{c^{\beta}}\left[\left(1+c^{\beta}\right) - \left(1+c^{\beta}\right)^{\frac{-r}{\beta}}\right] \quad (13)$$

The estimation for the parameters of DWD via maximum likelihood estimation technique presented the independent observations are $x_1, x_2, \dots, x_n$, then the likelihood function of the DWD shown:

$$\mathbf{L}\,(\boldsymbol{\beta}, \boldsymbol{c}; \boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}) = \sum_{i=1}^{n} \boldsymbol{ln}(\mathbf{g}(\mathbf{x_i}; \boldsymbol{\beta}, \boldsymbol{c}))$$

$$L(\beta,c) = n\,ln\,(\beta) + n ln\ (1+c^{\beta}) - n\,\beta\ln c + (\beta-1)\sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n} x_i^{\beta} + \sum_{i=1}^{n} ln\,(1-e^{-x_i^{\beta}c^{\beta}}) \quad (14)$$

This admits the partial derivatives:

$$\frac{\partial L(\beta,c)}{\partial\beta} = \frac{n}{\beta} + \frac{nc^{\beta}logc}{1+c^{\beta}} - nlnc + \sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n} x_i^{\beta}\, lnx_i + \sum_{i=1}^{n}\frac{e^{-x_i^{\beta}c^{\beta}}(x_i^{\beta}lnx_ic^{\beta} + x_i^{\beta}c^{\beta}lnc)}{1-e^{-x_i^{\beta}c^{\beta}}} \quad (15)$$

$$\frac{\partial L(\beta,c)}{\partial c} = \frac{n\beta c^{\beta-1}}{1+c^{\beta}} - \frac{n\beta}{c} + \sum_{i=1}^{n}\frac{e^{-x_i^{\beta}c^{\beta}}\beta x_i^{\beta}c^{\beta-1}}{1-e^{-x_i^{\beta}c^{\beta}}} \quad (16)$$

Equating these equations to zero

$$\frac{n}{\beta} + \frac{nc^{\beta}lnc}{1+c^{\beta}} - nlnc + \sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n} x_i^{\beta}\, lnx_i + \sum_{i=1}^{n}\frac{e^{-x_i^{\beta}c^{\beta}}(x_i^{\beta}lnx_ic^{\beta} + x_i^{\beta}c^{\beta}lnc)}{1-e^{-x_i^{\beta}c^{\beta}}} = 0 \quad (17)$$

$$\frac{n\beta c^{\beta-1}}{1+c^{\beta}} - \frac{n\beta}{c} + \sum_{i=1}^{n}\frac{e^{-x_i^{\beta}c^{\beta}}\beta x_i^{\beta}c^{\beta-1}}{1-e^{-x_i^{\beta}c^{\beta}}} = 0 \quad (18)$$

For estimation of parameters which may solve simultaneously for $\hat{\boldsymbol{\beta}}$ **and** $\hat{\mathbf{c}}$ ***and*** Variance covariance matrix given below

$$\mathbf{H(x)} = \begin{pmatrix} \frac{\partial^2(\mathbf{L}\,(\,((\mathbf{g}(\mathbf{X};,\boldsymbol{\beta},\mathbf{c}))}{(\partial\boldsymbol{\beta})^2} & \frac{\partial^2(\mathbf{L}\,(\,((\mathbf{g}(\mathbf{X},\boldsymbol{\beta},\mathbf{c}))}{(\partial\boldsymbol{\beta}\partial\mathbf{c})} \\ \frac{\partial^2(\mathbf{L}\,(\,((\mathbf{g}(\mathbf{X},\boldsymbol{\beta},\mathbf{c}))}{(\partial\mathbf{c}\partial\boldsymbol{\beta})} & \frac{\partial^2(\mathbf{L}\,(\,((\mathbf{g}(\mathbf{X},\boldsymbol{\beta},\mathbf{c}))}{(\partial\mathbf{c})^2} \end{pmatrix}$$

The inverse of the asymptotic covariance matrix given $\mathbf{I}(\boldsymbol{\lambda}, \mathbf{c}) = -\mathbf{E}(\mathbf{H}(\mathbf{X}))$ with

$$\frac{\partial^2(ln(g(X;,\beta,c))}{(\partial\beta)^2} = -\frac{n}{\beta^2} - \frac{c^{2\beta}nln[c]^2}{(1+c^{\beta})^2} + \frac{c^{\beta}nln[c]^2}{1+c^{\beta}} - \sum_{i=1}^{n} ln[x_i]^2x_i^{\beta} + \sum_{i=1}^{n}\Bigg(-\frac{e^{-2c^{\beta}x_i^{\beta}}\left(-c^{\beta}ln[c]x_i^{\beta} - c^{\beta}ln[x_i]x_i^{\beta}\right)^2}{\left(1-e^{-c^{\beta}x_i^{\beta}}\right)^2} - \frac{e^{-c^{\beta}x_i^{\beta}}\left(-c^{\beta}ln[c]x_i^{\beta} - c^{\beta}ln[x_i]x_i^{\beta}\right)^2}{1-e^{-c^{\beta}x_i^{\beta}}}$$
$$\frac{e^{-c^{\beta}x_i^{\beta}}\left(-c^{\beta}ln[c]^2x_i^{\beta} - 2c^{\beta}ln[c]ln[x_i]x_i^{\beta} - c^{\beta}ln[x_i]^2x_i^{\beta}\right)}{1-e^{-c^{\beta}x_i^{\beta}}} \quad \mathbf{(19)}$$

$$\frac{\partial^2(ln(g(X;\beta,c))}{(\partial c)^2} = \frac{n\beta}{c^2} + \frac{c^{-2+\beta}n(-1+\beta)\beta}{1+c^{\beta}} - \frac{c^{-2+2\beta}n\beta^2}{(1+c^{\beta})^2} + \sum_{i=1}^{n}\left(\frac{c^{-2+\beta}e^{-c^{\beta}x_i^{\beta}}(-1+\beta)\beta x_i^{\beta}}{1-e^{-c^{\beta}x_i^{\beta}}} - \frac{c^{-2+2\beta}e^{-2c^{\beta}x_i^{\beta}}\beta^2x_i^{2\beta}}{\left(1-e^{-c^{\beta}x_i^{\beta}}\right)^2} - \frac{c^{-2+2\beta}e^{-c^{\beta}x_i^{\beta}}\beta^2x_i^{2\beta}}{1-e^{-c^{\beta}x_i^{\beta}}}\right) \quad (20)$$

$$\frac{\partial^2(\ln g_3(X;\beta,c))}{(\partial\beta\partial c)} = -\frac{n}{c}+\frac{c^{-1+\beta}n}{1+c^\beta}-\frac{c^{-1+2\beta}n\beta\, ln[c]}{\left(1+c^\beta\right)^2}+\frac{c^{-1+\beta}n\beta\, ln[c]}{1+c^\beta}$$
$$+\sum_{i=1}^{n}\left(\frac{c^{-1+\beta}e^{-2c^\beta x_i^\beta}\beta x_i^\beta(-c^\beta\, ln[c]x_i^\beta-c^\beta ln[x_i]x_i^\beta)}{(1-e^{-c^\beta x_i^\beta})^2}+\frac{c^{-1+\beta}e^{-c^\beta x_i^\beta}\beta x_i^\beta\left(-c^\beta ln[c]x_i^\beta-c^\beta ln[x_i]x_i^\beta\right)}{1-e^{-c^\beta x_i^\beta}}-\frac{e^{-c^\beta x_i^\beta}\left(-c^{-1+\beta}x_i^\beta-c^{-1+\beta}\beta\, ln[c]x_i^\beta-c^{-1+\beta}\beta ln[x_i]x_i^\beta\right)}{1-e^{-c^\beta x_i^\beta}}\right) \quad (21)$$

$$\frac{\partial^2(ln(g_3(X;\beta,c)))}{(\partial c\partial\beta)} = -\frac{n}{c}+\frac{c^{-1+\beta}n}{1+c^\beta}-\frac{c^{-1+2\beta}n\beta ln[c]}{(1+c^\beta)^2}+\frac{c^{-1+\beta}n\beta ln[c]}{1+c^\beta}$$
$$+\sum_{i=1}^{n}\left(\frac{c^{-1+\beta}e^{-c^\beta x_i^\beta}x_i^\beta}{1-e^{-c^\beta x_i^\beta}}+\frac{c^{-1+\beta}e^{-c^\beta x_i^\beta}\beta ln[c]x_i^\beta}{1-e^{-c^\beta x_i^\beta}}+\frac{c^{-1+\beta}e^{-c^\beta x_i^\beta}\beta ln[x_i]x_i^\beta}{1-e^{-c^\beta x_i^\beta}}+\frac{c^{-1+\beta}e^{-2c^\beta x_i^\beta}\beta x_i^\beta\left(-c^\beta ln[c]x_i^\beta-c^\beta ln[x_i]x_i^\beta\right)}{\left(1-e^{-c^\beta x_i^\beta}\right)^2}+\frac{c^{-1+\beta}e^{-c^\beta x_i^\beta}\beta x_i^\beta\left(-c^\beta ln[c]x_i^\beta-c^\beta ln[x_i]x_i^\beta\right)}{1-e^{-c^\beta x_i^\beta}}\right) \quad (22)$$

RESULTS DISCUSSIONS

Several properties of the newly proposed distribution were obtained and studied numerically. Plots for the cumulative distribution function (CDF), probability density function (PDF), hazard function were also presented in this article. Moreover, inverse hazard, the information generating, moment generating functions were introduced and studied. In addition to these properties, Mills ratio, moments, and tables for the measure of skewness and kurtosis for selected parameter values were also provided. To estimate the unknown parameters, maximum likelihood method was used. A real data set has also been considered to show a practical application of the proposed distribution. The suggested distribution was found more appropriate with shape parameter (β) and scale parameter (c). The following graphical representations for Probability Density Function, Cumulative Distribution Function, Survival Function, Hazard Rate Function and Reverse Hazard Rate Function of DWD given below, with different shape and scale parameters.

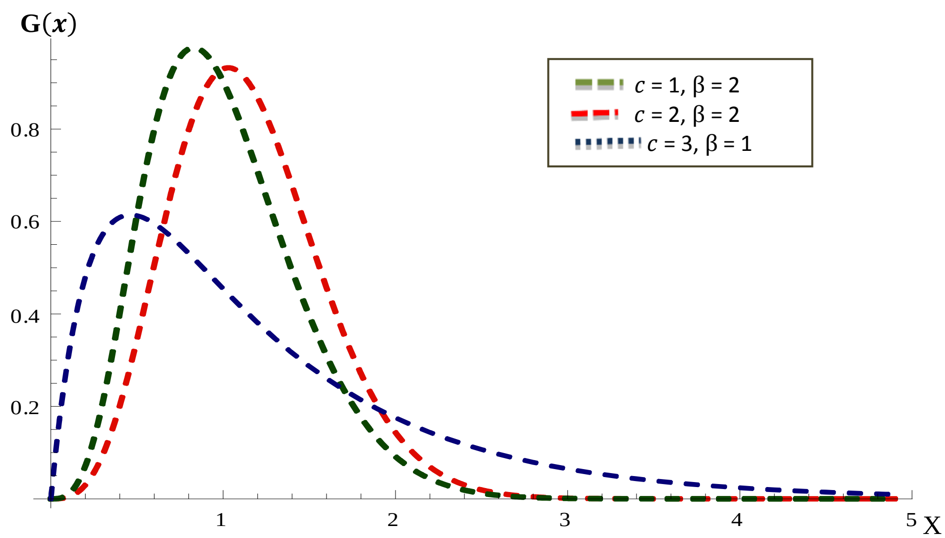

Fig. 1: Probability Density Function of Double Weibull Distribution for the Indicated Values of c and β

F (x)

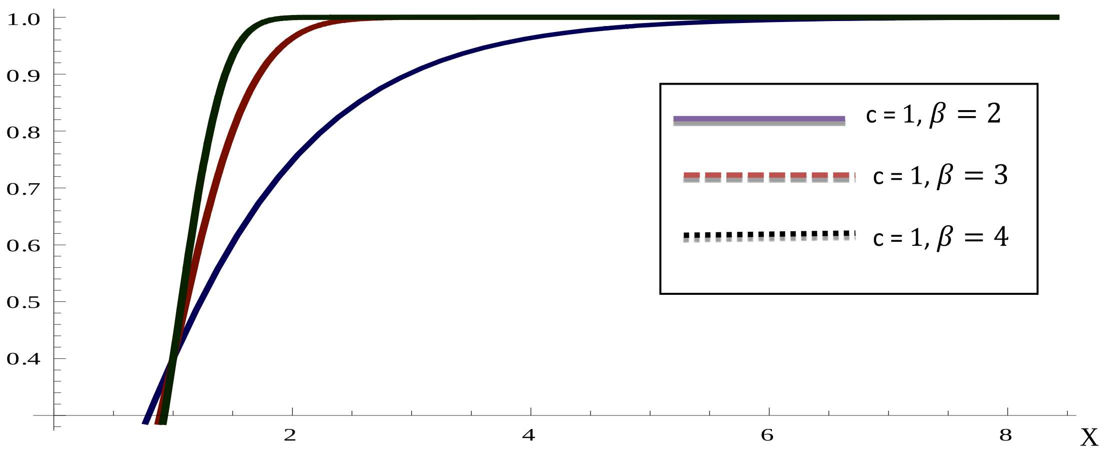

Fig 2: Cumulative Distribution Function of Double Weibull Distribution for the Indicated Values of c and β

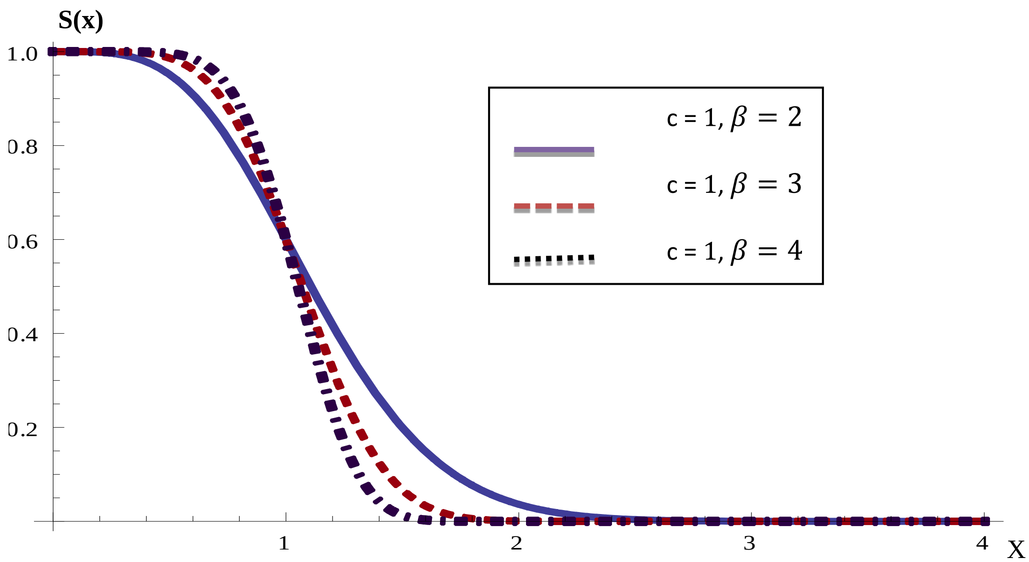

Fig 3: Survival Function of Double Weibull Distribution for the Indicated Values of c and β

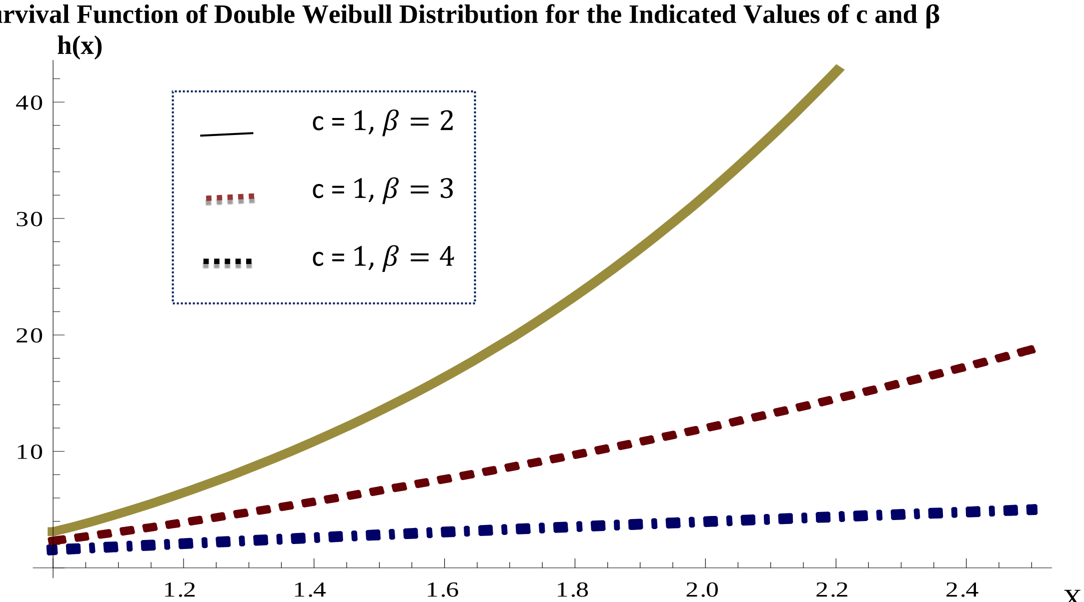

Fig 4: Hazard Rate Function of Double Weibull Distribution for the Indicated Values of c and β

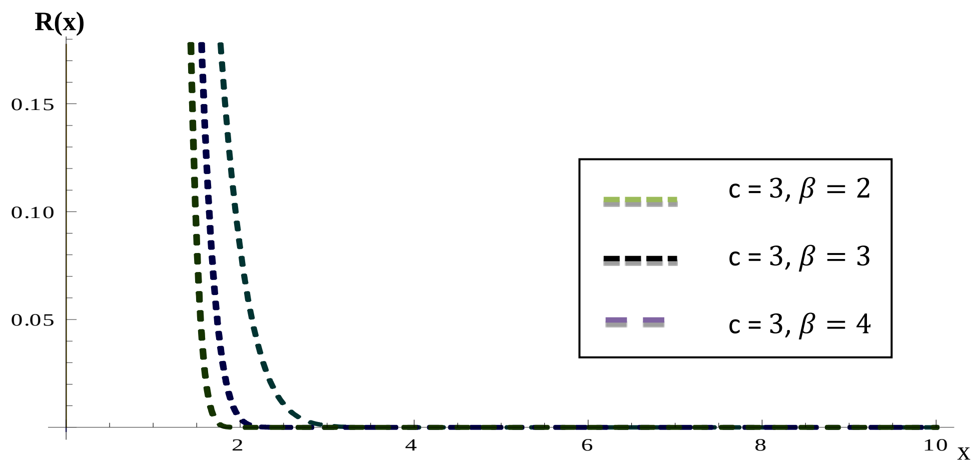

Fig.5: Reverse Hazard Rate Function of Double Weibull Distribution for the Indicated Values of c and β

To apply the proposed distribution on a real dataset, we considered the lifetime data of 20 electronic segments by Nasiru (2015).

Table 1: Maximum Likelihood Estimator's for Double Weibull Distribution and Weibull Distribution.

Distribution	Estimates	
	$\hat{c}$	$\hat{\beta}$
DWD	0.784991	0.667012
WD	-	0.805738

We estimated the Double Weibull distribution parameters $\hat{c}$ and $\hat{\beta}$ by using the Newton Raphson method. The estimated parameters were $\hat{\beta}$=0.667012 and $\hat{c}$ =0.784991. Moreover, we also compared our distribution to Weibull distribution using some well-known goodness of fit tests like Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson–Darling test (AD), Cramer-von Mises Test (CvM) and likelihood method (L). The results have been presented in the following table:

Table 2: Goodness of Fit tests.

Model	Goodness of fit criteria				
	AIC	BIC	L	AD	CvM
DWD	67.8094	69.8009	-31.9047	0.629024	0.0997923
WD	76.8506	77.8464	-37.4253	5.40742	1.04021

Table 3: Means and Standard Deviations of Double Weibull Distribution

C	β	Mean and Standard Deviation
0.5	2	1.2620∓0.4595
	3	1.1680∓0.0845
	4	1.1240∓0.2044
	5	1.0980∓0.1330
1	2	1.1450∓0.4326
	3	1.0770∓0.2783
	4	1.0500∓0.2054
	5	1.0370∓0.1638
2	2	1.0080∓0.4272
	3	0.9509∓0.2922
	4	0.9351∓0.2318
	5	0.9326∓0.1961
3	2	0.9530∓0.4368
	3	0.9152∓0.3084
	4	0.9141∓0.2467
	5	0.9200∓0.2071

Table 4: Coefficients of Skewness and Kurtosis of Double Weibull Distribution

C	β	$\sqrt{\beta_1}$	β_2
1	2.0	0.5010	3.240
2	2.0	0.6800	3.460
1	3.0	0.1440	3.010
1	3.5	0.0350	2.990
1	3.6	0.0170	2.980
1	3.7	0.0001	2.988
1	3.8	0.00001	2.999

The existing Weibull distribution was modified and named as Double Weibull Distribution (DWD). From the above table, it was observed that the DWD has smaller value of AIC as compared to the WD and hence better fit to real data sets. In addition to AIC, we also considered BIC test and DWD was found better than the WD. Moreover, some nonparametric tests, like Anderson–Darling test, Cramer-von Mises Test (CvM) and likelihood method (L) were also used and we found that DWD provide a better fit than the WD.

Waloddi Weibull who was introduced Wibull distribution first time, and many of the researchers worked on it and developed new horizons of the Weibull distribution. Basically the idea that we have discussed in the paper was taken from Area biased weighted weibull distribution introduced by (Perveen *et. al.*, 2016), A modified Weibull extension with bathtub-shaped failure rate function *by (*Xie *et. al.*, 2002) and MM double Exponential distribution by (Perveen and Munir, 2016). Our main objective was to develop a new, efficient Weibull distribution where we have developed double weight in existing probability density function of the Weibull distribution. Also, we acknowledged the extended of Weibull type distribution and finite mixture of distributions by (Al-Saleh and Agarwal, 2006) and double Weibull distribution by (Balakrishnan and Kocherlakota, 1985).

The hazard function may not be constant with respect to time in case of Weibull distribution that were discussed in “New hazard rate functions” (Dhillon, 1978).

Finally, the results conclude that the performance of the proposed distribution was remarkably good when we found several properties of the proposed distribution and satisfied all assumptions of probability distribution with application of real-life data set.

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REFERENCES

Al-Saleh, J. A., and S. K. Agarwal (2006). Extended Weibull type distribution and finite mixture of distributions. Stat. Meth., 3(3): 224-233.

Ghitany, M. E., E. K. Al-Hussaini, and R. A. Al-Jarallah (2005). Marshall–Olkin extended Weibull distribution and its application to censored data. J. App. Stat., 32(10): 1025-1034.

Lockhart, R. A., and M. A. Stephens (1994). Estimation and tests of fit for the three-parameter Weibull distribution. J. Royal. Stat. Soc. Sr. B (Meth.): 491-500.

Merovci, F., and I. Elbata, (2015). Weibull Rayleigh distribution: Theory and applications. App. Math. and Info. Sci., 9(4): 2127-2137.

Nadarajah, S., and A. K. Gupta (2006). Some bivariate gamma distributions. App. Math. Lett., 19(8): 767-774.

Nasiru, S. (2015). Another weighted weibull distribution from Azzalini's family. Euro. Sci. J., 11(9): 134-144.

Perveen, Z., M. Munir, Z. Ahmed, and M. Ahmad (2016). On area biased weighted weibull distribution. Sci. Int., 28(4): 3669-3679.

Perveen, Z., M. Munir (2016). MM Double Exponential Distribution. Int. J. Sci. and Eng. Res., 8(7): 720-728.

Reményi, N., and B. Vidakovic (2015). Wavelet shrinkage with double Weibull prior. Comm. Stat. Sim. and Comp., 44(1): 88-104.

Rinne, H. (2008). The Weibull Distribution: a handbook. CRC Press.

Seenoi, P., T. Supapakorn, and W. Bodhisuwan, (2014). The length-biased exponentiated inverted Weibull distribution. Int. J. Pure and App. Math., 92(2): 191-206.

Tang, Y., M. Xie, and T. N. Goh (2003). Statistical analysis of a Weibull extension model. Com. Stat. Theo. Meth., 32(5): 913-928.

Teimouri, M., and A. K. Gupta, (2013). On the three-parameter Weibull distribution shape parameter estimation. J. Data. Sci., 11(3): 403-414.

Xie, M., Y. Tang and T. N. Goh (2002). A modified Weibull extension with bathtub-shaped failure rate function. Rel. Eng. Sys. Safety. 76(3): 279-285.