Pakistan Journal of Science (Vol. 66 No. 3 September, 2014)

APPLICATION OF LAPLACE DECOMPOSITION METHOD FOR SOLVING SINE-GORDON EQUATION

Z. Zafar, A. M. Sultan*, A. Pervaiz**, M.O.Ahmed** and S. Hussain**

Faculty of Information Technology, University of Central Punjab Lahore, Pakistan.
*Department of Mathematics, University of South Asia Lahore, Pakistan.
**Department of Mathematics, University of Engineering & Technology Lahore, Pakistan.
Corresponding Author: zainulabadin1@hotmail.com

ABSTRACT: In this study we considered the initial value problem for the sine-Gordon equation by using Laplace Decomposition Method (LDM). The advantage of this method was its ability and flexibility to provide the analytical or approximate solutions to linear and nonlinear equations without linearization or discretization which makes it reliable for solving sine-Gordon equation.

Key words: Approximate solutions, Laplace Decomposition method, Adomian decomposition method, Sine Gordon Equation.

INTRODUCTION

The sine-Gordon equation which arises in the study of differential geometry of surfaces with Gaussian curvature has wide applications in the propagation of fluxon in Josephson junctions (Perring and Skyrme, *1962*, Whitham, *1999*, Sirendaoreji and Jiong, *2002*, Fu *et al., 2004*) between two superconductors, the motion of a rigid pendulum attached to a stretched wire, solid state physics, nonlinear optics, stability of fluid motions, dislocations in crystals and other scientific fields (Whitham, *1999*). Since its wide applications and important mathematical properties, many methods have been presented to study the different solutions and physical phenomena related to this equation (Bratsos and Twizell, *1998*, Kaya, *2003*, Wang, *2006*, Wei, *2000*, Wazwaz, *2006*, Peng, *2003*). Reviewing these improvements, by means of linearization, discretization or transformations, this equation is transformed to more simple equation, and then different types of solutions are followed. Unlike the classical schemes, we will consider the sine-Gordon equation by a new approach in this study.

METHODOLOGY

In this section we established an Algorithm using Laplace Decomposition method on the partial differential equations which were nonlinear. We considered the general form of inhomogeneous nonlinear partial differential equations with initial conditions as given below

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \quad (1)$$

$$u(x,0) = f(x), u_t(x,0) = g(x) \quad (2)$$

where L is second order differential operator $L - \frac{\partial^2}{\partial t^2}$

R is the remaining linear operator, Nu represents a general nonlinear differential operator and $h(x,t)$ was source term. The first step we will take Laplace transform on equation (1).

$$\mathcal{L}[L\,u(x,t)] + \mathcal{L}[R\,u(x,t)] + \mathcal{L}[N\,u(x,t)] = \mathcal{L}[h(x,t)] \quad (3)$$

By applying the Laplace transform differentiation property, we have

$$s^2\mathcal{L}[u(x,t)] - sf(x) - g(x) + \mathcal{L}[Ru(x,t)] + \mathcal{L}[Nu(x,t)] = \mathcal{L}[h(x,t)] \quad (4)$$

$$\mathcal{L}[u(x,t)] = \frac{1}{s}f(x) + \frac{1}{s^2}g(x) + \frac{1}{s^2}\mathcal{L}[h(x,t)] - \frac{1}{s^2}\mathcal{L}[Ru(x,t)] - \frac{1}{s^2}\mathcal{L}[Nu(x,t)] \quad (5)$$

The 2nd step in LDM is that we signify the solution as an infinite series as

$$u = \sum_{n=0}^{\infty} u_n(x,t) \quad (6)$$

The nonlinear operator is written as

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(u) \quad (7)$$

where $A_n(u)$ is Adomian polynomial [4] of $u_0, u_1, u_2, \ldots, u_n$ and it can be calculated by formula given below

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, n = 0,1,2 \dots \quad (8)$$

Putting the equation (6), equation (7) and equation (8) in equation (5), which gave this result

$$\mathcal{L}\left[\sum_{n=0}^{\infty} u_n(x,t)\right] = \frac{1}{s}f(x) + \frac{1}{s^2}g(x) + \frac{1}{s^2}\mathcal{L}[h(x,t)] - \frac{1}{s^2}\mathcal{L}\left[R\sum_{n=0}^{\infty} u_n(x,t)\right] - \frac{1}{s^2}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n(u)\right]$$

(9)

Or

$$\left[\sum_{n=0}^{\infty}\mathcal{L}\{u_n(x,t)\}\right] = \frac{1}{s}f(x) + \frac{1}{s^2}g(x) + \frac{1}{s^2}\mathcal{L}[h(x,t)] - \frac{1}{s^2}\mathcal{L}[Ru(x,t)] - \frac{1}{s^2}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n(u)\right]$$

(10)

When we compare the left and right hand sides of equation (10) we obtained

$$\mathcal{L}[u_0(x,t)] = \frac{1}{s}f(x) + \frac{1}{s^2}g(x) + \frac{1}{s^2}\mathcal{L}[h(x,t)] \quad (11)$$

$$\mathcal{L}[u_1(x,t)] = -\frac{1}{s^2}\mathcal{L}[Ru_0(x,t)] - \frac{1}{s^2}\mathcal{L}[A_0(u)] \quad (12)$$

$$\mathcal{L}[u_2(x,t)] = -\frac{1}{s^2}\mathcal{L}[Ru_1(x,t)] - \frac{1}{s^2}\mathcal{L}[A_1(u)] \quad (13)$$

Thus the recursive relation, in general form is

$$\mathcal{L}[u_{n+1}(x,t)] = -\frac{1}{s^2}\mathcal{L}[Ru_n(x,t)] - \frac{1}{s^2}\mathcal{L}[A_n(u)] \quad (14)$$

Now apply inverse Laplace transform to Eq. (11) to Eq. (14), so our recursive relation is as follow

$$u_0(x,t) = K(x,t) \quad (15)$$

$$u_{n+1}(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_n(x,t)] + \frac{1}{s^2}\mathcal{L}[A_n(u)]\right] \quad (16)$$

where $K(x,t)$ represented the expression from source term and with the initial conditions. First we applied Laplace transform on the right hand side of Eq. (16) then by taking the inverse Laplace transform we obtained the values of $u_0, u_1, u_2, \dots, u_n$ correspondingly.

APPLICATION 1

Let us consider the homogeneous partial differential equation

$$u_{tt} = u_{xx} + \sin u \quad (17)$$

The initial conditions were

$$u(x,0) = \frac{\pi}{2}, u_t(x,0) = 0 \quad (18)$$

Applying the algorithm of Laplace transform on equation (17), we have

$$s^2\mathcal{L}[u(x,t)] - su(x,0) - u_t(x,0) = \mathcal{L}[u_{xx}(x,t)] + \mathcal{L}[\sin u] \quad (19)$$

Using given initial condition on Eq. (19) , we have

$$s^2\mathcal{L}[u(x,t)] = \frac{s\pi}{2} + \mathcal{L}[u_{xx}(x,t)] + \mathcal{L}[\sin u] \quad (20)$$

Or

$$u(x,s) = \frac{1}{s^2}\left(\frac{s\pi}{2}\right) + \frac{1}{s^2}\mathcal{L}[u_{xx}(x,t)] + \frac{1}{s^2}\mathcal{L}[\sin u] \quad (21)$$

Now by applying the inverse Laplace transform to Eq. (21), we got

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\left(\frac{\pi}{2}\right)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t)]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[\sin u]\right] \quad (22)$$

The LDM assumed a series of solutions of the function $u(x,t)$ which is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (23)$$

The nonlinear term was handled with the help of Adomian Polynomials as below

$$\sin x = \sum_{n=0}^{\infty} A_n(u) \quad (24)$$

Putting equation (23) and equation (24) in equation (22), we have

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\left(\frac{\pi}{2}\right)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n(u)\right]\right] \quad (25)$$

The recursive relation was as follow

$$u_0(x,t) = \frac{\pi}{2} \quad (26)$$

$$u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_0(x,t)\right] + \frac{1}{s^2}\mathcal{L}[A_0(u)]\right]$$

(27)

$$u_2(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_1(x,t)\right] + \frac{1}{s^2}\mathcal{L}[A_1(u)]\right]$$

(28)

Similarly we have

$$u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_n(x,t)\right] + \frac{1}{s^2}\mathcal{L}[A_n(u)]\right]$$

(29)

Also the nonlinear terms were of the form of

$$A_0 = \sin u_0$$

$$A_1 = u_1 \cos u_0$$

$$A_2 = u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0$$

$$A_3 = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0$$

…..

After substituting the value u_0 , it resulted as

$$A_0 = \sin u_0 = \sin\frac{\pi}{2} = 1$$

Now

$$u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}\frac{\pi}{2}\right] + \frac{1}{s^2}\mathcal{L}[1]\right]$$

$$u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[0] + \frac{1}{s^2}\mathcal{L}[1]\right]$$

$$u_1(x,t) = \mathcal{L}^{-1}\left[\frac{\frac{1}{s^2 . 1}}{s}\right] = \frac{t^2}{2!}$$

$$A_1 = \frac{t^2}{2!}\cos\frac{\pi}{2}$$

$$A_1 = 0$$

$$u_2(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}\left(\frac{t^2}{2!}\right)\right] + \frac{1}{s^2}\mathcal{L}[0]\right]$$

$$u_2(x,t) = 0$$

$$A_2 = 0.\cos\frac{\pi}{2} - \frac{1}{2!}\left(\frac{t^2}{2!}\right)^2 \sin\frac{\pi}{2}$$

$$A_2 = -\frac{1}{2!}\frac{t^4}{4} = -\frac{t^4}{8}$$

$$u_3(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_2(x,t)\right] + \frac{1}{s^2}\mathcal{L}[A_2(u)]\right]$$

$$u_3(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}0\right] + \frac{1}{s^2}\mathcal{L}\left[-\frac{t^4}{8}\right]\right]$$

$$u_3(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[-\frac{t^4}{8}\right]\right]$$

$$u_3(x,t) = -\frac{t^6}{240}$$

The first few terms of $u_n(x,t)$ follow immediately

$$u(x,t) = \frac{\pi}{2} + \frac{t^2}{2!} + 0 - \frac{t^6}{240}$$

$$u(x,t) = \frac{\pi}{2} + \frac{t^2}{2!} - \frac{t^6}{240}$$

Similarly, $u_3, u_4, u_5 \cdots$ etc. can be obtained. Hence, all components of the decomposition (LDM) were identified. The complete solution is

$$u = \sum_{n=0}^{\infty} u_n$$

$$u(x,t) = \frac{\pi}{2} + \frac{t^2}{2!} - \frac{t^6}{240} + \cdots$$

This was our required result in Series form, which was the same as obtained in Partial Differential Equations and Solitary Waves Theory (Wazwaz, *2009*).

APPLICATION 2

Let us consider the homogeneous partial differential equation

$$u_{tt} = u_{xx} + \sin u \quad (30)$$

The initial condition was

$$u(x,0) = \frac{\pi}{2}, u_t(x,0) = 1 \quad (31)$$

Applying the algorithm of Laplace transform on equation (30), we got

$$s^2\mathcal{L}[u(x,t)] - su(x,0) - u_t(x,0) = \mathcal{L}[u_{xx}(x,t)] + \mathcal{L}[\sin u]$$

(32)

Using given initial condition on Eq. (30), it becomes

$$s^2\mathcal{L}[u(x,t)] = \frac{s\pi}{2} + 1 + \mathcal{L}[u_{xx}(x,t)] + \mathcal{L}[\sin u]$$

(33)

Or

$$u(x,s) = \frac{1}{s^2}\left(\frac{s\pi}{2}\right) + \frac{1}{s^2} + \frac{1}{s^2}\mathcal{L}[u_{xx}(x,t)] + \frac{1}{s^2}\mathcal{L}[\sin u]$$

(34)

Now by applying the inverse Laplace transform to Eq. (34), we got

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\left(\frac{\pi}{2}\right)+\frac{1}{s^2}\right]+\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t)]\right]+\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[\sin u]\right]$$

(35)

The LDM assumed a series of solution of the function $u(x,t)$ which is given below

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (36)$$

The nonlinear term was handling with the aid of Adomian Polynomials as below

$$\sin x = \sum_{n=0}^{\infty} A_n(u) \quad (37)$$

Putting equation (36) and equation (37) in equation (35), we have

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\left(\frac{\pi}{2}\right)+\frac{1}{s^2}\right]+\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right]\right]+\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n(u)\right]\right] \quad (38)$$

The recursive relation was as follows

$$u_0(x,t) = \frac{\pi}{2}+t \quad (39)$$

$$u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_0(x,t)\right]+\frac{1}{s^2}\mathcal{L}[A_0(u)]\right]$$

(40)

$$u_2(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_1(x,t)\right]+\frac{1}{s^2}\mathcal{L}[A_1(u)]\right]$$

(41)

Similarly we have

$$u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_n(x,t)\right]+\frac{1}{s^2}\mathcal{L}[A_n(u)]\right]$$

(42)

Also the nonlinear terms were

$$A_0 = \sin u_0$$
$$A_1 = u_1 \cos u_0$$
$$A_2 = u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0$$
$$A_3 = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0$$

…..

After substituting the value u_0, it resulted as

$$A_0 = \sin u_0 = \sin\left(\frac{\pi}{2}+t\right)$$
$$A_0 = \sin u_0 = \cos t$$

Now

$$u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\pi}{2}+t\right)\right]+\frac{1}{s^2}\mathcal{L}[\cos t]\right]$$
$$u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[0]+\frac{1}{s^2}\mathcal{L}[\cos t]\right]$$
$$u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(\frac{s}{s^2+1}\right)\right]$$
$$u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\left(\frac{1}{s^2+1}\right)\right]$$
$$u_1(x,t) = (1-\cos[t])$$
$$A_1 = (1-\cos[t])\cos\left(\frac{\pi}{2}+t\right)$$
$$A_1 = (1-\cos[t])(-\sin t)$$
$$A_1 = (-\sin t + \sin t\cos[t])$$

$$u_2(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\frac{\partial^2}{(\partial x^2)(1-\cos[t])}\right]+\frac{1}{s^2}\mathcal{L}[-\sin t+\sin t\cos t]\right]$$
$$u_2(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[0]+\frac{1}{s^2}\mathcal{L}[-\sin t+\sin t\cos t]\right]$$
$$u_2(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[-\sin t+\sin t\cos t]\right]$$
$$u_2(x,t) = \sin t - \frac{3}{4}t - \frac{1}{8}\sin 2t$$

The first few terms of $u_n(x,t)$ follows immediately

$$u(x,t) = \frac{\pi}{2}+t+(1-\cos[t])+\sin t-\frac{3}{4}t-\frac{1}{8}\sin 2t$$
$$u(x,t) = 1+\frac{\pi}{2}+\frac{1}{4}t-\cos t+\sin t-\frac{1}{8}\sin 2t$$

Similarly, $u_3, u_4, u_5 \cdots$ etc. could be obtained. Hence, all components of the decomposition (LDM) were identified. The complete solution was

$$u = \sum_{n=0}^{\infty} u_n$$
$$u(x,t) = 1+\frac{\pi}{2}+\frac{1}{4}t-\cos t+\sin t-\frac{1}{8}\sin 2t+\cdots$$

So the series solution resulted as

$$u(x,t) = \frac{\pi}{2}+t+\frac{1}{2}t^2-\frac{1}{4}t^4+\cdots$$

obtained upon Taylor expansion for the Trigonometric functions involved. This was our required result in Series form, as obtained in Partial Differential Equations and Solitary Waves Theory (Wazwaz, *2009*).

RESULTS AND DISCUSSION

In this work we have applied Laplace Decomposition Method to Sine Gordon equation. A series of solutions of the Sine Gordon equation were developed by using this method and an efficient result has been obtained. The Laplace Decomposition method was a powerful tool to search for solutions of various linear and nonlinear initial value problems. In the real world problems we dealt with ambiguous conditions for example the uncertainties and vagueness in the values of the initial conditions. This method overcame these difficulties. A comparison of this method with Adomian Decomposition revealed healthy similar results. We observed that the working of the method was simple and straight forward as it has been checked by two applications. The method gave more realistic series solutions that converged very rapidly in physical problems.

REFERENCES

Bratsos, A.G., and E.H. Twizell, A family of parametric finite-difference methods for the solution of the sine-Gordon equation, Appl. Math. Comput. 93:117-137 (1998).

Fu, Z., S. Liu and S. Liu, Exact solutions to double and triple sinh-Gordon equations, Z Naturforsch 59: 933-937 (2004).

Kaya, D., A numerical solution of the sine-Gordon equation using the modified decomposition method, Appl. Math. Comput. 143:309-317(2003).

Peng, Y.Z., Exact solutions for some nonlinear partial differential equations, Phys. Lett. A, 314, 401-408 (2003).

Perring, J.K.and T.H. Skyrme, A model unified field equation, Nucl. Phys.31:550-555 (1962).

Sirendaoreji and S. Jiong, A direct method for solving sinh-Gordon type equation, Phys. Lett. A , 298:133-139 (2002).

Wang, Q., An application of the modified Adomian decomposition method for (*N*+1)-dimensional Sine-Gordon field, Appl. Math. Comput. 181:147-152 (2006).

Wazwaz,A.M., Exact solutions for the generalized sine-Gordon and the generalized sinh-Gordonequations, Chaos, Solitons & Fractals 28, 127-135 (2006).

Wazwaz, A. M., *Partial Differential Equations and Solitary Waves Theory*, Higher Education Press, Beijing (2009).

Wei, G.W., Discrete singular convolution for the sine-Gordon equation, Phys. D, 137:247-259 (2000).

Whitham, G.B., Linear and nonlinear waves, Wiley-Interscience, New York (1999).