FIXED POINTS OF MULTI-VALUED A^*-MAPPINGS IN CONE METRIC SPACE.

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ABSTRACT: In this paper, we extend the idea of A^*-mappings from metric spaces to cone metric spaces. Further, we prove some fixed point theorems for multi-valued mapping using A^*-contractions in cone metric space setup. Our results generalizes andimproves some results in the existing literature from metric spacessetup to cone metric spaces setup. Furthermore, some fixed point theorems in cone metics spaces are also improved.

Key words:Cone metric, Fixed points,self-maps, Multi-valued mappings, A^*-mappings.

INTRODUCTION

The concept of cone metric spacewas introduced by (Huang and Zhang 2007) replacing the set of positive real numbers by an ordered Banach space. They obtained some fixed point theorems in cone metric spaces using the normality of the cone. The results of (Huang and Zhang 2007) were generalized by (Rezapour and Haghi 2008). Several other authers like (Aage and Salunke 2009), (Ilic and Rakocevic 2008) and (Abbas and Rhoades2009) were also investigating some common fixed point theorems for different types of contractive mappings in cone metric space. The idea of general multi-valued A^*-maps was introduced by (Akram et al. 2003) and they proved some fixed point theorems for these maps in complete metric spaces. The purpose of this paper is to proved the existence of fixed point of a general class of multi-valued maps called A^*-maps in cone metric spaces.

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Preliminaries: Let E be a topological vector space. A subset P of E is called a cone if and only if:

1. P is closed, nonempty and $P \neq 0$;
2. $a, b \in R, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;
3. $P \bigcap (-P) = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is said to be normal space E if there ia a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

The least positive number satisfying the above inequality is called the normal constant of P, while $x << y$ stands for $y - x \in intP$ (interior of P). (Rezapour and Hamlbarani Haghi 2008) proved that there is no normal cone with normal constant $K < 1$ and for each $k > 1$ there is a cone with normal constant $K > k$.

Definition 2.1 *Let* X *be a nonempty set. Suppose that the mapping* $d : X \times X \rightarrow E$ *satisfies:*

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of cone metric space more general than metric space.

Definition 2.2 *Let* (X, d) *be a cone metric space,* $\{x_n\}$ *a sequence in* X *and* $x \in X$. *For every* $c \in E$ *with* $0 << c$, *we say that* $\{x_n\}$ *is*

(i) a Cauchy sequence if there is a natural number N such that for all $n, m > N$, $d(x_n, x_m) << c$.

(ii) a convergent sequence if there is a natural number N such that for all $n > N$, $d(x_n, x_m) << c$ for some $x \in X$

It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 2.3 *A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.*

Definition 2.4 *A set A in a cone metric space X is closed if for every sequence $\{x_n\}$ in a A which converges to some $x \in X$ implies that $x \in A$.*

Let X be a cone metric space. We denote by $P(X)$ the family of all nonempty subsets of X, and by $P_{cl}(X)$ the family of all nonempty closed subsets of X. A point x in X is called a fixed point of a multi-valued mapping $T : X \to P_{cl}(X)$ provided $x \in Tx$. The collection of all fixed points of T is denoted by $F(T)$.

On the other hand, Akram et al. (2003) defined A^*-contraction as follows: Let a nonempty set A^* consisting of all functions $\alpha : R_+^3 \to R_+$ satisfying

(i) α is continuous on the set R_+^3 of all triplets of nonnegative reals(with respects to the Euclidean metric on R^3).

(ii) α is non-decreasing in each coordinate variable;

(iii) $a \le kb$ for some $k \in [0,1)$, whenever $a \le \alpha(a,b,b)$ or $a \le \alpha(b,a,b)$ or $a \le \alpha(b,b,a)$ for all a,b.

Definition 2.5 *A multiple-valued mapping $T : X \to P(X)$ is said to be A^*-contractions. If*

$$\delta(Tx,Ty) \le \alpha(\delta(x,y), \delta(x,Tx), \delta(y,Ty)) \quad (1)$$

for some $\alpha \in A^*$ and for all $x, y \in X$. We also call these mappings as A^*-mappings.

Throughout the sequel, $CB(X)$ would denote the set of all closed and bounded subsets of X, where X is a Complete cone metric space. For sets A and B in a cone metric space X, we use the symbols,

$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}$,

$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$ and

$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$.

Main Result for multi-valued maps

Theorem 3.1 *Let (X,d) be a Complete cone metric space and $T_1, T_2 :\to P_{cl}(X)$ be a multi-valued mappings suchthat for $i, j \in \{1,2\}$ with $i \ne j$, for each $x, y \in X$, $u_x \in T_i(x)$ and $u_y \in T_j(y)$ such that*

$$d(u_x,u_y) \le \alpha(d(x,y), d(x,u_x), d(y,u_y)). \quad (2)$$

Then $F(T_1) = F(T_2) \ne \emptyset$ and $F(T_1) = F(T_2) \in P_{cl}(X)$.

Proof. Suppose that x_0 is an arbitrary point of X. For $i, j \in 1,2$ with $i \ne j$, Then $x_1 \in T_i(x_0)$. There exist $x_2 \in T_j(x_1)$ such that

$$d(x_1,x_2) \le \alpha(d(x_0,x_1), d(x_0,x_1), d(x_1,x_2))$$
$$\le k\, d(x_0,x_1),$$

where $k \in [0,1)$. Similarly,

$$d(x_2,x_3) \le k\, d(x_1,x_2) \le k^2\, d(x_0,x_1).$$

Continuing in a similar way, we get

$$d(x_n,x_{n+1}) \le k\, d(x_{n-1},x_n) \le .... \le k^n\, d(x_0,x_1), \forall n \ge 1.$$

Then for $m > n$; we get

$$d(x_n,x_m) \le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... + d(x_{m-1},x_m)$$

$$\le (k^n + k^{n+1} + k^{m-1})\, d(x_0,x_1)$$
$$= k^n d(x_0,x_1)/(1-k).$$

Let $c >> 0$ be given, choose a symmetric open neighborhood V of 0 such that $c + V \subseteq P$. Also, choose a natural number N_1 such that

$$k^n\, d(x_0,x_1)/(1-k) \in V, \forall n \ge N_1. \quad (3)$$

Which implies that $k^n\, d(x_0,x_1)/(1-k) << c$ for all $n > N_1$ and hence $d(x_n,x_m) << c$ for all $m,n > N_1$. Hence $\{x_n\}$ be a Cauchy sequence in X, since X is complete cone metric space, there exist an element $x \in X$ s.t. $x_n \to x$ as $n \to \infty$. Let $0 << c$ be given and for $x_{2n} \in T_j(x_{2n-1})$, there exist $u_n \in T_i(x)$, such that

$$d(x_{2n},u_n) \le \alpha(d(x_{2n-1},x), d(x_{2n-1},x_{2n}), d(x,u_n)).$$

Taking limits $n \to \infty$, we get

$$\lim_{n\to\infty} d(x_{2n},u_n) \le \lim_{n\to\infty} [\alpha(d(x_{2n-1},x), d(x_{2n-1},x_{2n}), d(x,u_n))]$$

$$d(x, \lim_{n\to\infty} u_n) \le \alpha(d(x,x), d(x,x), d(x, \lim_{n\to\infty} u_n)$$
$$= \alpha(0,0, d(x, \lim_{n\to} u_n))$$
$$\le k(0) = 0.$$

This implies that $\lim_{n\to\infty} u_n = x$ Hence $u_n \to x$ as $n \to \infty$.

Since $T_i(x)$ is closed, $x \in F(T_i)$ and $F(T_i) \ne \emptyset$.

Let $x \in X$ be a fixed point of T_1 Then by hypothesis, there exists $w \in T_2(x)$ s.t.

$$d(w,x) \le \alpha(d(x,x), d(x,x), d(x,w)$$
$$\le \alpha(0,0, d(x,w))$$
$$\le k(0) = 0.$$

and so, $x = w$, Thus $F(T_1) \subset F(T_2)$, similarly, $F(T_2) \subset F(T_1)$. Now we prove that $F(T_i)$ is closed. Let $\{x_n\}$ be a Cauchy sequence in $F(T_j) = F(T_i)$ such that $x_n \to x$ as $n \to \infty$. Since $x_n \in T_i(x_{n-1})$, there exits $v_n \in T_j(x)$ such that

$$d(x_n, v_n) \le \alpha(d(x_{n-1}, x), d(x_{n-1}, x_n), d(x, v_n))$$
$$\le \alpha(d(x_{n-1}, x), d(x_{n-1}, x_n), d(x, v_n)).$$

Taking limit $n \to \infty$, we get

$$d(x, lim_{n\to\infty} v_n) \le \alpha(d(x,x), d(x,x), d(x, lim_{n\to\infty} v_n)$$
$$\le \alpha(0,0, d(x, lim_{n\to\infty} v_n))$$
$$\le k(0) = 0.$$

This implies that $v_n \to x$ as $n \to \infty$. Since $v_n \in T_j(x)$ for each $n \in N$ and $T_j(x)$ is closed $x \in T_j(x)$. Hence, $x \in F(T_j) = F(T_i)$ as required.

Theorem 2 of (Abbas, Rhoades and Nazir 2009) on cone metric space become the corollary of Theorem 3.1 as follows.

Corollary 3.2 *Let* (X,d) *be a Complete cone metric space and* $T_1, T_2 : X \to P_{cl}(X)$ *be a multi-valued mappings such that for* $i, j \in \{1,2\}$ *with* $i \ne j$, *for each* $x, y \in X$, $u_x \in T_i(x)$ *and* $u_y \in T_j(y)$,

$$d(u_x, u_y) \le a\, d(x,y) + b\, d(x, u_x) + c\, d(y, u_y), \quad (4)$$

where $a, b, c \ge 0$ are fixed constants with $a + b + c < 1$. Then $F(T_1) = F(T_2) \ne \emptyset$ and $F(T_1) = F(T_2) \in P_{cl}(X)$.

The following corollary extends Theorem 4.1 of (Latif and Beg 1997) to the case of two mappings on cone metric spaces.

Corollary 3.3 *Let* (X,d) *be a Complete cone metric space and* P *is a non-normal cone. If* $T_1, T_2 : X \to P_{cl}(X)$ *are two multi-valued mappings such that for* $i, j \in \{1,2\}$ *with* $i \ne j$, *for each* $x, y \in X$, $u_x \in T_i(x)$ *and* $u_y \in T_j(y)$,

$$d(u_x, u_y) \le h\big(d(x, u_x) + d(y, u_y)\big), \quad (5)$$

Then $F(T_1) = F(T_2) \ne \emptyset$ and $F(T_1) = F(T_2) \in P_{cl}(X)$.

The following corollary extends Theorem 4.1 of (Latif and Beg 1997) to cone metric spaces.

Corollary 3.4 *Let* (X,d) *be a Complete cone metric space and* P *is a non-normal cone. If* $T : X \to P_{cl}(X)$ *is a multi-valued mapping such that for each* $x, y \in X$, $u_x \in T(x)$ *and* $u_y \in T(y)$ *such that*

$$d(u_x, u_y) \le h\big(d(x, u_x) + d(y, u_y)\big), \quad (6)$$

where $0 \le h < 1/2$. Then $F(T) \ne \emptyset$ and $F(T) \in P_{cl}(X)$.

Corollary 3.5 *Let* (X,d) *be a Complete cone metric space and* P *is a non-normal cone. If* $T : X \to P_{cl}(X)$ *is a multi-valued mapping such that for each* $x, y \in X$, $u_x \in T(x)$ *and* $u_y \in T(y)$ *such that*

$$d(u_x, u_y) \le a\, d(x,y), \quad (7)$$

where $0 \le a < 1/2$. Then $F(T) \ne \emptyset$ and $F(T) \in P_{cl}(X)$.

Corollary 3.6 *Let* (X,d) *be a Complete cone metric space and* P *is a non-normal cone. If* $T : X \to P_{cl}(X)$ *is a multi-valued mapping such that for each* $x, y \in X$, $u_x \in T(x)$ *and* $u_y \in T(y)$ *such that*

$d(u_x,u_y) \le a\, d(x,y) + b\, d(x,u_x) + c d(y,u_y),$ (8)

where $a,b,c \ge 0$ are fixed constants with $a+b+c<1$. Then T has a fixed point.

Theorem 3.7 *Let* (X,d) *be a complete cone metric space and mappings* $T_1,T_2 : X \to CB(X)$ *satisfying the following conditions,* for each $x \in X$, $T_1(x), T_2(x) \in CB(X)$,

$H(T_1(x),T_2(y)) \le \alpha(d(x,y), d(x,T_1x), d(y,T_2y))$

where $\alpha \in A^*$, then there exists $p \in X$ such that $p \in T_1(x) \bigcap T_1(y)$.

Proof. Let $x_0 \in X$, $T_1(x_0)$ is non-empty closed bounded subset of X. Choose $x_1 \in T_1(x_0)$, for this x_1 by the same reason $T_2(x_1)$ is non-empty closed bounded subset of X.

$d(x_1,x_2) \le H(T_1(x_0),T_2(x_1))$
$\le \alpha(d(x_0,x_1), d(x_0,T_1(x_0)), d(x_1),T_2(x_1))$
$\le \alpha(d(x_0,x_1), d(x_0,x_1), d(x_1,x_2))$
$\le k\, d(x_0,x_1)$

where $k \in [0,1)$. For this x_2, $T_1(x_0)$ is a non-empty closed bounded subset of X. Since $x_2 \in T_2(x_1)$ and $T_2(x_1)$ and $T_1(x_2)$ both are closed and bounded subset of X, there exist $x_3 \in T_1(x_2)$ such that

$d(x_2,x_3) \le H(T_2(x_1),T_1(x_2)) = H(T_1(x_2),T_2(x_1))$
$\le \alpha(d(x_2,x_1), d(x_2,T_1(x_2)), d(x_1,T_2(x_1)))$
$\le \alpha(d(x_2,x_1), d(x_2,x_3), d(x_1,x_2))$
$= \alpha(d(x_1,x_2), d(x_2,x_3), d(x_1,x_2))$
$\le k\, d(x_1,x_2) \le k^2\, d(x_0,x_1).$

On continuing this process, we get a sequence $\{x_n\}$ such that $x_{n+1} \in T_2(x_n)$ or $x_{n+1} \in T_1(x_n)$ and $d(x_{n+1},x_n) \le k^n\, d(x_0,x_1).$

Let $0 << c$ be given, choose a natural number N_1 such that $k^n d(x_0,x_1) << c, \forall n \ge N_1$. This implies that $d(x_n,x_{n+1}) << c.$ Therefore, $\{x_n\}$ is a Cauchy sequence in (X,d) is a complete cone metric space, there exist $p \in X$ s.t. $x_n \to p.$

$d(T_1(p),p) \le d(p,x_n) + d(x_n,T_1(p))$
$\le d(p,x_n) + H(T_2(x_{n-1}),T_1(p))$
$\le d(p,x_n) + H(T_1(p),T_2(x_{n-1}))$
$\le d(p,x_n) + \alpha(d(p,x_{n-1}), d(p,T_1(p)), d(x_{n-1},x_n)).$

Taking $\lim_{n\to\infty}$, we get

$d(T_1(p),p) \le d(p,p) + \alpha(d(p,p), d(p,T_1(p)), d(p,p))$

$\le 0 + \alpha(0, d(T_1(p)), 0)$
$\le k(0) = 0.$

This implies that $d(T_1(p),p) = 0$, or $p \in T_1(p)$. Similarly, $p \in T_2(p)$. Hence $p \in T_1(p) \bigcap T_2(p)$ Which is required.

Theorem 3.8 *Let* X *be a cone complete metric space and* $T : X \to CB(X)$ *a mapping satisfying*

$H(Tx,Ty) \le \alpha(d(x,y), d(x,Tx), d(y,Ty))$

if there exists an asymptotically T-regular sequence $\{x_n\}$ in X. Then T has a fixed point x^* in X. Moreover, $Tx_n \to Tx^*$.

Proof. Let $x_0 \in X$, consider $x_{n+1} \in Tx_n$, n=0,1,2,…. Now, consider

$d(x_{n+2},x_{n+1}) \le H(Tx_{n+1},Tx_n)$
$\le \alpha(d(x_{n+1},x_n), d(x_{n+1},Tx_{n+1}), d(x_n,Tx_n))$
$\le \alpha(d(x_{n+1},x_n), d(x_{n+1},x_{n+2}), d(x_n,x_{n+1}))$
$\le k\, d(x_{n+1},x_n)$

where $k \in [0,1)$. But

$d(x_{n+2},x_{n+1}) \le H(Tx_{n+1},Tx_n) \le k\, d(x_{n+1},x_n)$.

This implies that

$H(Tx_{n+1},Tx_n) \le k\, d(x_{n+1},x_n)$.(9)

Similarly,

$d(x_{n+3},x_{n+2}) \le k\, d(x_{n+2},x_{n+1}) \le k^2 d(x_{n+1},x_n)$

for $m > n$, we get

$d(x_m,x_n) \le d(x_m,x_{m+1}) + d(x_{m+1},x_{m+2}) + ... + d(x_{n-1},x_n)$
$\le (k^m + k^{m-1} + ... + k^{n-1}) d(x_0,x_1).$

Hence $\{x_n\}$ be a Cauchy sequence in X. By equation(9), we get $\{Tx_n\}$ is a Cauchy sequence. Since $(CB(X),H)$ is a complete cone metric space. There exists a $K^* \in CB(X)$, s.t. $H(Tx_n, K^*) \to 0.$ Let $x^* \in K^*$ s.t. $x_n \to x^*$. Then

$$d(x^*,Tx^*) \le H(K^*,Tx^*) = \lim_{n\to\infty} H(Tx_n,Tx^*)$$
$$\le \lim_{n\to\infty} (\alpha(d(x_n,x^*), d(x_n,Tx_n), d(x^*,Tx^*)))$$
$$\le \lim_{n\to\infty} (\alpha(d(x_n,x^*), d(x_n,x_{n+1}), d(x^*,Tx^*)))$$
$$\le \alpha(d(x^*,x^*), d(x^*,x^*), d(x^*,Tx^*))$$
$$\le k(0) = 0.$$

This implies that $d(x^*,Tx^*) = 0$ or $x^* \in Tx^*$. Now

$$H(K^*,Tx^*) = \lim_{n\to\infty} H(Tx_n,Tx^*) = 0.$$

This implies that

$$Tx^* = K^* = \lim_{n\to\infty} Tx_n.$$

Corollary 3.9 *Let* X *be complete metric space and* $T: X \to CB(X)$ *a mapping satisfying*

$$H(Tx,Ty) \le \alpha_1(d(x,Tx))d(x,Tx) + \alpha_2(d(y,Ty))\, d(y,Ty)$$

for all $x, y \in X$, where $\alpha_i : R \to [0,1)$, $i = 1,2$. If there exists an asymptotically T-regular sequence $\{x_n\}$ in X. Then T has fixed point x^* in X. Moreover, $Tx_n \to Tx^*$.

Conclusion: In this paper,we extend the idea of A^*-mappings from metric spaces to cone metric spaces. We, extended and improved severalresults of (Abbas,Rhoads and Nazir 2009) for A*-mappings. Further, we generalised and improved fiew results of (Latif and Beg 1997) for A*-mappings in cone metric spaces setup.

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