POLYNOMIAL CUBIC SPLINE METHOD FOR SOLVING FOURTH-ORDER PARABOLIC TWO POINT BOUNDARY VALUE PROBLEMS

A. Pervaiz and M. O. Ahmad

Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan.
Corresponding Author:anjumpervaiz1@hotmail.com

ABSTRACT: Partial differential equations (PDEs) played a vital role in natural sciences and engineering. Different studies were carried out for solving fourth-order partial differential equations. These equations governed the oblique vibrations of a uniform beam. This study presented and illustrated a new technique for the numerical approximations of fourth-order partial differential equations (PDEs). The new technique was based on the fact of employing polynomial cubic spline method (PCSM) along with the Adomian decomposition method (ADM). The Adomian decomposition method was used to obtain the boundary conditions for the replaced variables, whereas continuous approximation was constructed and applied on the decomposed system of PDEs. The performance of the developed scheme was illustrated by numerical tests that involved numerical approximations with the exact solutions on a collection of test problems.

Keywords: Fourthorder parabolic partial differential equations, Adomian decomposition method, Polynomial cubicspline technique, Finite difference approximations, Continuous approximation.

(Received 12-04-2014 Accepted 20-03-15)

INTRODUCTION

The fourth- order parabolic PDEs of the form,
$u_{tt} + u_{xxxx} = h(x,t), 0 \leq x \leq 1, t > 0$,(1)
withinitial conditions (ICs),
$u(x,0) = f(x), u_t(x,0) = g(x)$, (2)
and boundary conditions (BCs) at $x = 0$ and $x = 1$,
$u(0,t) = \alpha(t), u(1,t) = \beta(t)$, (3)

arise in the physical phenomena of undamped transverse vibrations of a flexible straight beam in such a way that its support does not contribute to the strain energy of the system (Shahid and Arshed, 2013).In equation (1),u represents the transverse displacement of the beam, t and x are time and displacement variables, respectively, and $h(x,y)$ is dynamic driving force per unit mass. While, $f(x)$, $g(x)$, $\alpha(t)$, and $\beta(t)$ are continuous functions(Evans and Yousif, 1991).

Many researchers who have worked on equation (1) i.e.,(Collatz,1973; Crandall,1954; Conte and Royster,1956; Conte,1957; Evans,1965; Jain *et al.*,1976;Richtmyerand Mortan, 1967) by using finite differences and reducing equation (1) in a system of second orderpartial differential equations. An explicit and implicit finite difference method based on the semi-explicit method by (Lees, 1961) and high accuracy method by (Douglas, 1956) is derived by (Fairweather and Gourlay,1967). Three-time-level scheme based on parametric quintic spline is presented by (Aziz *et al.*, 2005). Caglar and Caglar, (2008) used fifth degree B-spline for the solution of fourth-order parabolic partial differential equations and produced accurate results. In a study (Evans and Yousif, 1991) used alternating group explicit method (AGE) for solving fourth-order parabolic PDEs. A power series solution is suggested by (Wazwaz, 1995, 2001) using Adomian decomposition method. Rashidinia (1994) had studied homogeneous problem. (Khan *et al.*, 2005) proposed sextic spline solution for the solution of fourth-order parabolic PDEs. (Shahidand Arshed, 2013) presented a quintic B-spline solution for equation (1). (Fengnan *et al*., 2014) considered a non-linear model describing crystal surface growth and described a finite element method for a non-linear parabolic equation.

MATERIALS AND METHODS

The primary focus of this paper was to develop a new technique for the numerical approximations of fourth-order partial differential equations. In this case, equation (1) was decomposed into a system of PDEs as follows

Let $u_{xx} = v$ and $u_t = w \implies u_{tt} = w_t$ .Then equation (1) reduced to the form
$w_t + v_{xx} = h(x,t)$. (4)

The corresponding decomposed system of partial differential equations became
$v_{xx} = -w_t + h(x,t)$,(5)
$u_{xx} = v$,(6)
$u_t = w$,(7)

Provided the initial and boundary conditions given in equations (2) and (3), respectively. Adomian decomposition method was used to obtain the boundary conditions for the changed variables v and w .For the fourth-order parabolic equations (Wazwaz, 2009)

$u_0(x,t) = f(x) + tg(x)$.(8)
Assume$v(x,t) = u_{0xx}(x,t) = f''(x) + tg''(x)$,(9)
and$w(x,t) = u_{0t}(x,t) = g(x)$.(10)
Then the boundary conditions for v and $w$were of the form
$v(0,t) = u_{0xx}(0,t) = f''(0) + tg''(0)$,(11)
$v(1,t) = u_{0xx}(1,t) = f''(1) + tg''(1)$,(12)
$w(0,t) = u_{0t}(0,t) = g(0)$,(13)
$w(1,t) = u_{0t}(1,t) = g(1)$. (14)
Equations (5-7) along with the initial conditions in equation (2) and boundary conditions in equations (3), and (11-14) form a system of partial differential equations.

Construction of Polynomial Cubic Spline Method(PCSM): In a study Ahlberg*et al.*,(1967) showed that cubic spline $S(x)$ interpolating the function$u(x)$ at grid points:$x_i = x_0 + ih(i = 0,1,\cdots,n)$over the interval $x_{i-1} \le x \le x_i$wasgiven by the equation
$S(x) = L_{i-1}\frac{(x_i-x)^3}{6h} + L_i\frac{(x-x_{i-1})^3}{6h} + \left(u_{i-1} - \frac{h^2}{6}L_{i-1}\right)\frac{(x_i-x)}{h} + \left(u_i - \frac{h^2}{6}L_i\right)\frac{(x-x_{i-1})}{h}$, (15)
where,$L_i = S''(x_i)$ and $u_i = u(x_i)$.
Hence
$S(x_i+) = -\frac{h}{3}L_i - \frac{h}{6}L_{i+1} + \frac{u_{i+1}-u_i}{h}$, $(i = 0,1,\cdots,n-1)$,(16)
and
$S(x_i-) = \frac{h}{3}L_i + \frac{h}{6}L_{i-1} + \frac{u_i-u_{i-1}}{h}$, $(i = 1,2,\cdots,n)$,(17)
so that the continuity of the first derivative implied
$\frac{6}{h}L_{i+1} + \frac{2h}{3}L_i + \frac{6}{h}L_{i-1} = \frac{u_{i+1}-2u_i+u_{i-1}}{h}$, $(i = 1,2,\cdots n-1)$.(18)
Thus for equations (5) and (6) with the associated boundary conditions (3) and (11-14), the requirements that the spline approximation should satisfy the partial differential equations (5) and (6) at the grid points:x_i $(i = 0,1,\cdots,n)$. This lead on to using equations (16) and (17) to a set of relationships (18), from which L_i, $i = 0,1,\cdots,n$, could be eliminated.
For the demonstration of the method developed above, equations (5) and (6) at the grid points (x_i,u_i), (x_i,v_i) and (x_i,w_i)were discretized as follows
Approximating$w_t \cong \frac{w_i-w_{i-1}}{k}$,(19)
and$u_t \cong \frac{u_i-u_{i-1}}{k}$,(20)
in equations (5) and (7), respectively, we had
$v_i'' = -\left(\frac{w_i-w_{i-1}}{k}\right) + h(x_i,t)$,(21)
$u_i'' = v_i$,(22)
$\frac{u_i-u_{i-1}}{k} - w_i = 0$.(23)
Take$w_{i-1} \cong g(x_i) = g_i$,$u_{i-1} \cong f(x_i) = f_i$, and replaced by$v_i''$ by L_i and u_i'' by M_i.Thenequations (21), (22), and (23), respectively, were as under
$L_i = -\frac{1}{k}(w_i - g_i) + h(x_i,t)$,(24)
$M_i = v_i$,(25)
$u_i - kw_i - f_i = 0$.(26)
Now, equations (24) and (25) could be written as
$L_{i+1} = -\frac{1}{k}(w_{i+1} - g_{i+1}) + h(x_{i+1},t)$,(27)
$L_{i-1} = -\frac{1}{k}(w_{i-1} - g_{i-1}) + h(x_{i-1},t)$,(28)
$M_{i+1} = v_{i+1}$,(29)
$M_{i-1} = v_{i-1}$.(30)

Similar results, as of equation (18), could be written for equations (5) and (6) as under
$L_{i-1} + 2L_i + L_{i+1} = \frac{6}{h^2}(w_{i+1} - 2w_i + w_{i-1})$,(31)
$M_{i-1} + 2M_i + M_{i+1} = \frac{6}{h^2}(v_{i+1} - 2v_i + v_{i-1})$.(32)

Substituting the values of L_i, L_{i+1} and L_{i-1} from equations (24), (27), and (28) into the equation (31) and the values of M_i, M_{i+1} and M_{i-1} from equations (25), (29), and (30) in equation (32), we had
$\left(\frac{6}{h^2} + \frac{1}{k}\right)w_{i+1} + \left(-\frac{12}{h^2} + \frac{4}{k}\right)w_i + \left(\frac{6}{h^2} + \frac{1}{k}\right)w_{i-1} = \frac{1}{k}(g_{i+1} + 4g_i + g_{i-1}) + h(x_{i+1},t) + 4\,h(x_i,t) + h(x_{i-1},t)$, and(33)
$v_{i-1} + 4v_i + v_{i+1} = \frac{6(v_{i-1}-2v_i+v_{i+1})}{h^2}$.(34)

Thus equations (26), (33), and (34) associated with the boundary conditions in equations (3), and (11-14) form a complete system of algebraic equations, which could be solved by using simple numerical techniques.

RESULTS AND DISCUSSIONS

To quantify the quality of the above developed technique, we considered the following test problems.

Test Problem 1: We Consideredthe following fourth-order parabolic boundary value problem (BVP)
$u_{tt} + u_{xxxx} = 0, 0 < x < \pi\ ,\ t > 0$,
withI. Cs
$u(x,0) = Cos\ x = f_i$,$u_t(x,0) = -Sin\ x = g_i$,
And B.Cs
$u(0,t) = Cos\ t$,$u(\pi,t) = -Cos\ t$.

The analytical solution to the above BVP was
$u(x,t) = Cos\ (x+t)$.

The first set of experiments was performed to observe the absolute error while comparing the polynomial cubic spline method with the exact solution applied to the above test problem. The associated absolute errors for $h = \frac{1}{5}$ and $k = 0.1$ had been shown in Table 1. Where we observed in Table 1 that, for different values of spatial displacement x, the maximum difference between the exact and numerically approximated solution was not more than 1.1×10^{-02}. For the same set of experiments as listed in Table 1, we performed experiments at different time step levels, i.e., $k = 0.01$ and $k = 0.001$ as shown in Table 2 and Table 3, respectively. The overall conclusion was that smaller the time step level more the numerical accuracy. The best observed smallest absolute error, numerical accuracy was approximately 8.3×10^{-07}.

Table1: Showing absolute errors at $h = \frac{1}{5}$, $k = 0.1$

x	Exact	PCSM	Absolute Error
0.0	0.995004165	0.995004165	0.0000E+00
0.4	0.746294669	0.743464885	2.8298E-03
0.8	0.212525975	0.202951453	9.5745E-03
1.2	-0.402420418	-0.415082536	1.2662E-02
1.6	-0.863655889	-0.874569104	1.0913E-02
2.0	-0.995004165	-0.995004165	0.0000E+00

Table 2: Showing absolute errors at $h = \frac{1}{5}$, $k = 0.01$

x	Exact	PCSM	Absolute Error
0.0	0.99995	0.99995	0.0000E+00
0.4	0.803098789	0.803077771	2.1018E-05
0.8	0.299491137	0.299407129	8.4008E-05
1.2	-0.31851195	-0.31862686	1.1491E-04
1.6	-0.814854298	-0.814956218	1.0192E-04
2.0	-0.99995	-0.99995	0.0000E+00

Table 3: Showing absolute errors at For $h = \frac{1}{5}$, $k = 0.001$

x	Exact	PCSM	Absolute Error
0.0	0.9999995	0.9999995	0.0000E+00
0.4	0.808428805	0.808428601	2.0358E-07
0.8	0.308065784	0.308064954	8.2941E-07
1.2	-0.309967896	-0.309969035	1.1384E-06
1.6	-0.809604375	-0.809605388	1.0126E-06
2.0	-1	-1	0.0000E+00

Test Problem 2: While considered the following fourth-order parabolic boundary value problem (BVP)
$u_{tt} + u_{xxxx} = (\pi^4 - 1)Sin\ \pi x\ Sin\ t, 0 < x < \pi\ , t > 0,$
with the I.Cs
$u(x,0) = 0 = f_i, u_t(x,0) = Sin\ \pi x = g_i,$
andB.Cs
$u(0,t) = 0, u(\pi,t) = Sin\ \pi^2\ Sin\ t.$
The analytical solution to the above BVP was
$u(x,t) = Sin\ \pi x\ Sin\ t.$

For the comparison of the polynomial cubic spline method with the exact solution, we repeated the same set of experiments for the test problem 2 as shown in Tables 4-6. It has been observed that the best observed numerical accuracy as obtained by the combination of $h = \frac{1}{5}, k = 0.001$, and $x = 0.8$.

Table 4: Showing absolute errors at $h = \frac{1}{5}$, $k = 0.1$

x	Exact	PCSM	Absolute Error
0.0	0	0	0.0000E+00
0.4	0.05868061	0.056543098	2.1375E-03
0.8	0.094947221	0.091488655	3.4586E-03
1.2	0.094947221	0.091488655	3.4586E-03
1.6	0.05868061	0.056543098	2.1375E-03
2.0	0	0	0.0000E+00

Table 5: Showing absolute errors at $h = \frac{1}{5}$, $k = 0.01$

x	Exact	PCSM	Absolute Error
0.0	0	0	0.0000E+00
0.4	0.005877755	0.005873432	4.3230E-06
0.8	0.009510407	0.009503412	6.9948E-06
1.2	0.009510407	0.009503412	6.9948E-06
1.6	0.005877755	0.005873432	4.3230E-06
2.0	0	0	0.0000E+00

Table 6: Showing absolute errors at $h = \frac{1}{5}$, $k = 0.001$

x	Exact	PCSM	Absolute Error
0.0	0	0	0.0000E+00
0.4	0.000587785	0.000587781	4.3676E-09
0.8	0.000951056	0.000951049	7.0669E-09
1.2	0.000951056	0.000951049	7.0669E-09
1.6	0.000587785	0.000587781	4.3676E-09
2.0	0	0	0.0000E+00

Wazwaz, (2001) discussed the analytic behavior for variable coefficient fourth order parabolic PDEs. Noise term phenomenon was implemented to obtain the numerical solution of non-homogeneous problem. Khan *et al.*, (2005) used sextic spline approximation for the numerical solution of fourth order parabolic PDE. They evaluated the problem at $h = 0.05$ with time steps $t = 10, 16, 25, 48, 64, 75, 100$. Fengnan *et al.*, (2014) exploited the non-linear parabolic eqution through finite element method. The error was calculated by employing L^2 norm at $h = \frac{\pi}{100}$ and $\Delta t = \frac{1}{100}$. Caglar and Caglar, (2008) tested B-spline on fourth order parabolic PDEs and the maximum absolute error was reported to be4.1×10^{-7} at $n = 721, k = 0.005$ .While, Evan and Yousif, (1965), Aziz *et al.*, (2005) and Shahid and Arshed, (2013) used $h = 10^{-1}, 16^{-1}, 20^{-1}, 25^{-1}, 75^{-1}, 100^{-1}$ number of step size to solve fourth order parabolic equations.Whereas, with the application of the current technique the results obtained were showing almost the same accuracy by using step size $h = 5^{-1}$. This demonstration gave the clear evidence that the method developed in this work performed nicely for the problems under consideration. The results obtained with the proposed method were very much close to the exact solutions as shown in the Tables 1-6. In Tables 1-3, it could be seen that the minimum error occurred close to the beginning grid points while in Tables 4-6, minimum error was occurred evenly at all the grid points.

REFERENCES

AhlbergJ. H., E. N. Nilson and J. L. Walsh. The Theory of Splines and their Applications. Academic Press, New York (1967).

Aziz T., A. Khan and J. Rashidinia. Spline methods for the solution of fourth order parabolic partial differential equations. Appl. Math. Comput. 167: 153-166 (2005).

Caglar H. and N. Caglar. Fifth-degree B-spline solution for a fourth-order parabolic partial differential equation. Appl. Math. Comput., 201: 597-603 (2008).

Collatz L. Hermitian methods for initial value problems in partial differential equations. J.J.H. Miller (Ed.), Topics in Numerical Analysis, Academic Press, pp. 41–61 (1973).

Conte S. D. A stable implicit finite difference approximation to a fourth order parabolic equation. J. Assoc. Comput. Mech. 4: 18–23 (1957).

ConteS.D. and W.C. Royster. Convergence of finite difference solutions to a solution of the equation of the vibrating rod. Proc. Am. Math. Soc. 7: 742–749 (1956).

Crandall S. H. Numerical treatment of a fourth order partial differential equations, J. Assoc. Comput. Mech. 1: 111–118 (1954).

Douglas J. The solution of the diffusion equation by a high order correct difference equation, J. Math. Phys. 35: 145–151 (1956).

Evans D. J. A stable explicit method for the finite difference solution of a fourth order parabolic partial differential equation. Comput. J. 8: 280–287 (1965).

Evans D. J. and W. S. Yousif. A note on solving the fourth order parabolic equation by the AGE method. Int. J. Comput. Math. 40: 93–97 (1991).

Fairweather G. and A.R. Gourlay. Some stable difference approximations to a fourth order parabolic partial differential equation. Math. Comput. 21: 1–11 (1967).

Fengnan L., Z. Xiaopeng and L. Bo.Finite element analysis of a nonlinear parabolic equation modeling epitaxial thin-film growth. Springer Boundary Value Problems, 2014:46 (2014).

JainM.K., S.R.K. Iyengar and A.G. Lone. Higher order difference formulas for a fourth order parabolic partial differential equation. Int. J. Numer. Methods Eng. 10: 1357–1367 (1976).

Khan A., I. Khan and T. Aziz. Sextic spline solution for solving fourth-order parabolic partial differential equation. Int. J. Comput. Math. 82: 871–879 (2005).

Lees M. Alternating direction and semi explicit difference methods for parabolic partial differential equations. Numer. Math. 3: 398–412 (1961).

Rashidinia J. Applications of splines to the numerical solution of differential equations, Ph.D Thesis, A. M. U. Aligarh (1994).

Richtmyer R.D. and K.W. Mortan. Difference Methods for Initial Value Problems. Second ed., Interscience, New York. (1967).

Shahid S. S. and S. Arshed. Quintic B-Spline for the Numerical Solution of Fourth-Order Parabolic Partial Differential Equations. World Applied Sci. J. 23: 115–122 (2013).

Wazwaz A.M. On the solution of the fourth order parabolic equation by the decomposition method. Int. J. Comput. Math. 57: 213–217 (1995).

Wazwaz A.M. Analytic treatment for variable coefficient fourth order parabolic partial differential equations. Appl. Math. Comput. 123: 219–227 (2001).

Wazwaz A.M. Partial Differential Equations and Solitary Waves Theory. Springer Dordrecht Heidelberg London New York (2009).