INDIRECT PANEL METHOD FOR IDEAL FLOW AROUND A SYMMETRIC AEROFOIL WITH CONSTANT VARIATION

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ABSTRACT: In this paper, the indirect panel method is used to compute the velocity distribution over the surface of a symmetric aerofoil for which the exact result is present. To check the accuracy of the method; the computed flow velocity is compared to the exact result for the flow over the surface of a symmetric aerofoil.

Key words: Indirect Panel Method, Ideal Flow, Symmetric Aerofoil, Constant Variation.

INTRODUCTION

The Indirect panel method is used to calculate flow past a symmetric aerofoil. This method depends on the distribution of singularities, such as sources or doublets, over the surface of the body and computes the unknowns in the form of singularity strengths. The equation of the indirect panel method can be derived using an approach based on Green's theorem. (Ramsay [1942], Lamb [1932], Milne-Thomson [1968], Shah [2008])

The importance of the panel methods have been widely recognized during the last few decades. The panel methods are presented under different names such as "boundary element methods", "surface singularity methods", boundary integral solution methods", or "boundary integral equations methods". This method has been successfully applied in several fields, for examples, elasticity, potential theory, aerodynamics, elasto-statics and elasto-dynamics etc. (Brebbia [1978], Brebbia and Walker [1980], Hess and Simth [1967], Morino et al [1975], Luminita et al [2008], Mushtaq et al[2009], Mushtaq and Shah [2010a, 2010b,2012] , Mushtaq [2011], Muhammad [2011])

Indirect panel methods are numerical techniques based on simplifying assumptions about the physics and characteristics of the ideal flow around a body. The assumptions of the problem are steady, irrotational, and ideal flow cause great simplifications to the general equations of aerodynamics. For an irrotational flow

$$\vec{V} = -\nabla\Phi \qquad (1)$$

Where is the velocity of the fluid at any point of the flow field and is the total velocity potential. But equation of continuity for steady and ideal flow is

$$\nabla.\vec{V} = 0 \qquad (2)$$

From equations (1) and (2), we get

$$\nabla^2 \varsigma.$$

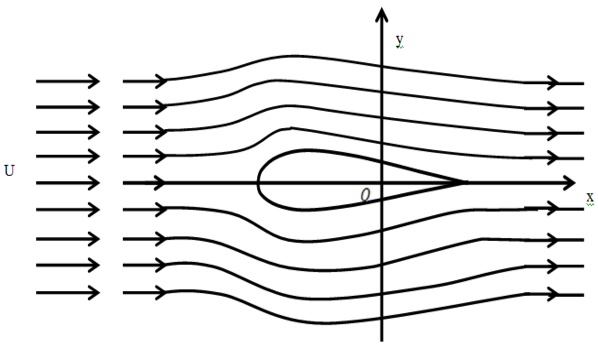
This is Laplace's equation that is an extensively studied linear partial differential equation.

For solution of the problem of ideal flow around arbitrary body, Laplace's equation can be solved using the boundary condition that there is no flow can penetrate the boundary of the body. Also the flow away from the body should be uniform.

The solution of this linear partial differential equation can be expressed in terms of an integral around the surface of the body using the vector identity. This integral equationconsists of expressions leading to boundary distribution of singular solution of a Laplaces' equation. Thus, the linear expression of a singular solution to Laplaces' equation also leads to a solution of a differential equation.

The simple solution technique for panel methods consist of discretizing the boundary of the body with straight panels and choosing singularities to be distributed around the panels in a particular way, but with unknown singularity-strength parameters. As every singularity is a solution to Laplaces' equation, a linear expression of the singular solution also leads to a solution of a differential equation. The ideal flow boundary condition should be satisfied at a distinct number of points said to becollocation points. This method results in a system of linear algebraic equations which can be solved for the unknown singularity-strength parameters. Descriptions of the method differ depending on the singularities involved and other descriptions of problem formulation. Thus the final result is repeatedly a system of linear algebraic equations which can be solved for the unknown singularity-strength parameters.

Flow Past a Symmetric Aerofoil: For an exterior ideal flow problem, Let the uniform stream with velocity in the +ve axis direction passes over a symmetric aerofoil as shown in the figure.



Exact Velocity: The modulus of the velocity distribution over the surface of a symmetric aerofoil (Chow[1979], Mushtaq and Shah [2010a, 2010b], and Mushtaq [2011]) is

$$V = U \left| \ldots \right| \qquad (3)$$

where

is the radius of the circular cylinder (C.C) and is the constant of Joukowski transformation and $a = b - r$ coordinate is the center of the circular cylinder.

Thus, in rectangular coordinates, equation (3) becomes as

$$V = U\sqrt{\frac{[\{(x-a)^2+y^2\}^2 - r^2\{(x-a)^2-y^2\}]^2 + 4r^4y(x-a)^2}{[(x-a)^2+y^2]^2}} \times \sqrt{\frac{[(x^2+y^2)^2 - b^2(x^2-y^2)]^2 + 4b^4x^2y^2}{(x^2+y^2)^2 - 2b^2(x^2-y^2) + b^4}} \qquad (4)$$

Boundary Condition of Aerofoil: For an ideal fluid flow, the boundary condition is

$$\vec{V}\cdot\hat{n} = 0 \qquad (5)$$

Where is the outward drawn unit normal to the surface of the symmetric aerofoil.

For an irrotational motion

$$\vec{V} = -\nabla\Phi \qquad (6)$$

Where is the total velocity potential.

Thus from equations (5) and (6),we have

$$-\nabla\Phi\cdot\hat{n} = 0$$

Which can also be written as

$$\frac{\partial\Phi}{\partial n} = 0 \qquad (7)$$

But

$$\Phi = \phi_{u.s} + \phi_{s.a} \qquad (8)$$

Differentiating equation (8), we have

$$\frac{\partial\Phi}{\partial n} = \frac{\partial\phi_{u.s}}{\partial n} + \frac{\partial\phi_{s.a}}{\partial n} \qquad (9)$$

Thus from equations (7) and (9), we get

$$\frac{\partial\phi_{u.s}}{\partial n} + \frac{\partial\phi_{s.a}}{\partial n} = 0$$

$$\frac{\partial\phi_{s.a}}{\partial n} = -\frac{\partial\phi_{u.s}}{\partial n} \qquad (10)$$

Since given in Thomson [1968], Shah [2008], Muhammad [2011], Mushtaq et al [2009], Mushtaq and Shah [2010a,2010b,2012], Mushtaq [2011], the velocity potential of the uniform stream is

$$= -Ux \qquad (11)$$

Then

$$\frac{\partial\phi_{u.s}}{\partial n} = -U\frac{\partial x}{\partial n} = -U(\hat{n}\cdot\hat{\imath}) \qquad (12)$$

Thus from equations (10) and (12), we get

$$\frac{\partial\phi_{s.a}}{\partial n} = U(\hat{n}\cdot\hat{\imath}) \qquad (13)$$

In general, the outward drawn unit normal to any vector

$$\vec{P} = (x_b - x_a)\hat{\imath} + (y_b - y_a)\hat{\jmath}$$

is given as

$$\hat{n} = \frac{-(y_b - y_a)\hat{\imath} + (x_b - x_a)\hat{\jmath}}{\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}}$$

Thus

$$\hat{n}\cdot\hat{\imath} = \frac{-(y_b - y_a)}{\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}} \qquad (14)$$

From equations (13) and (14), we get

$$\frac{\partial\phi_{s.a}}{\partial n} = U\left(\frac{-(y_b - y_a)}{\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}}\right)$$

$$= \left(\frac{-(y_b - y_a)}{\sqrt{(x_b - x_a)^2 + (y_b}}\right. \text{(taking)} \qquad (15)$$

This is the boundary condition for the symmetrical aerofoil.

Discritization of Indirect Panel Method Equation: For exterior ideal flow, two dimensional problems are computed by panel method. Thus the end points coordinates of the panels for the process of discritization of the surface of symmetric aerofoil are given as follows. The surface of the circular cylinder is divided into S elements in the counter clockwise position by applying the formula (Mushtaq et al [2009], Mushtaq and Shah [2010a, 2010b], Mushtaq [2011], and Muhammad[2011]).

$$\theta_j = [(S+3) - 2j]\frac{\pi}{S}, \quad j = 1,2,\ldots,S \qquad (16)$$

Then the end points of these S panels of circular cylinder are

$$\left.\begin{aligned}\xi_j &= -a + r\cos\theta_j\\ \eta_j &= r\sin\theta_j\end{aligned}\right\} \qquad (17)$$

Thus by applying Joukowski's transformation, the end points of the symmetric aerofoil are given as

$$z_j = \zeta_j + \frac{b^2}{\zeta_j} \qquad (18)$$

Hence if $\zeta_j =$ and $z_j =$ then from equation (18), we get

$$x_j + iy_j = \xi_j + i\eta_j + \frac{b^2}{\xi_j + i\eta_j}$$

$$x_j + iy_j = \xi_j + i\eta_j + \frac{b^2(\xi_j - i\eta_j)}{\xi_j^{\,2} + \eta_j^{\,2}}$$

$$x_j + iy_j = \left(\xi_j + \frac{b^2\xi_j}{\xi_j^{\,2} + \eta_j^{\,2}}\right) + i\left(y_j - \frac{b^2\eta_j}{\xi_j^{\,2} + \eta_j^{\,2}}\right)$$

$$x_j + iy_j = \left(1 + \frac{b^2}{\xi_j^{\,2} + \eta_j^{\,2}}\right)\xi_j + i\left(1 - \frac{b^2}{\xi_j^{\,2} + \eta_j^{\,2}}\right)\eta_j$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} x_j &= \left(1 + \frac{b^2}{\xi_j^{\,2} + \eta_j^{\,2}}\right)\xi_j \\ y_j &= \left(1 - \frac{b^2}{\xi_j^{\,2} + \eta_j^{\,2}}\right)\eta_j \end{aligned}\right\}, \quad j = 1,2,\ldots,S \qquad (19)$$

Matrix Formulation: The value of is supposed to be constant on every panel in the case of constant panel approach. This will be equal to the values at the mid-node of the panel. Thus, in this case, the number of nodes will be equivalent to the number of panels ‘S’. On every panel the variable is identified as a boundary condition. But is constant on every panel, it can be taken out of the integral. This gives

$$-c_i\Phi_i + \sum_{j=1}^{S}\left[\frac{1}{2\Pi}\int_{\Gamma_j - i}\frac{\partial}{\partial n}(\log\frac{1}{r})\,d\Gamma\right]\Phi_j = -(\Phi_{u.s})_i \qquad (20)$$

Equation (20) applies for a particular node ‘i’ and the integrals $\frac{1}{2\Pi}\int_{\Gamma_j - i}\frac{\partial}{\partial n}(\mathrm{l}c$ relate the node ‘i’ with the panel ‘j’ over which integrals are evaluated. These integrals will be denoted by Hence equation (20) can be written as

$$-c_i\Phi_i + \sum_{j=1}^{S}\hat{H}_{i,j}\Phi_j = -(\Phi_{u.s})_i \qquad (21)$$

The integral in equation (21) are difficult to evaluate analytically and are usually evaluated numerically using a 4-point Gauss Quadrature rule. The integrals over the panels will be calculated numerically.

Let

$$H_{i,j} = \begin{cases} \hat{H}_{i,j} & when\ i \neq j \\ \hat{H}_{i,j} - c_i & when\ i = j \end{cases} \qquad (22)$$

Equation (21) can be rewritten as

$$\sum_{j=1}^{S}\hat{H}_{i,j}\Phi_j = -(\Phi_{u.s})_i$$

The whole set of equations can be expressed in matrix form as

$$[H]\{\underline{U}\} = \{\underline{R}\} \qquad (23)$$

Where is matrix of influence coefficient, is a vector of unknown total potentials and on the R.H.S is a known vector whose panels are the negative of the values of the velocity potentials of the uniform stream at the nodes on the region of the body.

Assuming that the value of is given as a boundary condition on each panel of the region, then equation (23) has a set of ‘S’ unknowns. Equation (23) can be recorded in such a way that all the unknowns are on the L.H.S and can be then written as

$$[A]\{\underline{X}\} = \{\underline{R}\} \qquad (24)$$

Where is a vector of the unknown values of and is the coefficient matrix. The above equation represents a set of ‘S’ simultaneous linear equations in ‘S’ unknowns and can be solved by standard methods(Gauss Elimination Method, Gauss-Seidal Method) to give the values on the region. The values of at any point can be calculated from the equation (24) where boundary of the region is discretized into panels.

Exterior Flow Problems: For two-dimensional problems, the equation of indirect panel method in the case of a doublet distribution alone for exterior flow problems can be written as

$$-c_i\Phi_i + \frac{1}{2\Pi}\int_{\Gamma - i}\Phi\frac{\partial}{\partial n}(\log\frac{1}{r})\,d\Gamma + \Phi_\infty = -(\Phi_{us})_i \qquad (25)$$

For exterior flow problems, proceeding as in equation (21) and (25) for the indirect panel method can be expressed as

$$-c_i\Phi_i + \sum_{j=1}^{S}\hat{H}_{i,j}\Phi_j + \Phi_\infty = -(\Phi_{u.s})_i$$

or

$$\sum_{j=1}^{S} H_{i,j}\Phi_j + \Phi_\infty = -(\Phi_{u.s})_i$$

When all nodes are taken into consideration, equation (3.26) produces a $S \times$ system of equations which can put in the matrix form in case of constant panel as

$$[H]\{\underline{U}\} = \{\underline{R}\} \qquad (27)$$

Where is matrix of influence coefficient, is a vector of unknown total potentials and on the R.H.S is a known vector whose panels are the negative of the values of the velocity potentials of the uniform stream at the nodes on the region of the body. Note that in equation (27) has unknowns $\Phi_1 + \Phi_2 + \cdots \Phi$.To solve specially this system of equations, the value of at some position must be specified. For simplification is chosen as zero. This $S \times$ system reduces to an system of equations which can be solved as before but now the diagonal coefficients of will be found by the formula

$$H_{i,j} = -\sum_{\substack{j=1 \\ j\neq i}}^{S} H_{i,j} - 1 \qquad (28)$$

The velocity in the middle of two nodes on the boundary, can then be estimated by using the formula

$$\vec{V} = \frac{\Phi_{J+1} - \Phi_J}{\text{Length from node j to j} + 1} \qquad (29)$$

The calculated velocity distributions are compared with analytical solutions for the symmetric aerofoil using Fortran programming.

The following tables (1) to (4) and graphs (1) to (4) show the comparison of the computed velocities with exact velocity over the surface of a symmetric aerofoil for 8,16,32,and 64 panels using constant panel variation.

Table (1): Comparison of computed and exact velocity distribution over the boundary of symmetric aerofoil using for 8 constant panels.

ELEMENT	X	Y	R	VELOCITY	EXACT VELOCITY
1	-1.68	.33	1.72	.83222E+00	.96125E+00
2	-1.22	.78	1.45	.20056E+01	.19969E+01
3	-.57	.77	.96	.19847E+01	.16958E+01
4	-.10	.32	.33	.81904E+00	.55641E+00
5	-.10	-.32	.33	.81904E+00	.55641E+00
6	-.57	-.77	.96	.19847E+01	.16958E+01
7	-1.22	-.78	1.45	.20056E+01	.19969E+01
8	-1.68	-.33	1.72	.83221E+00	.96125E+00

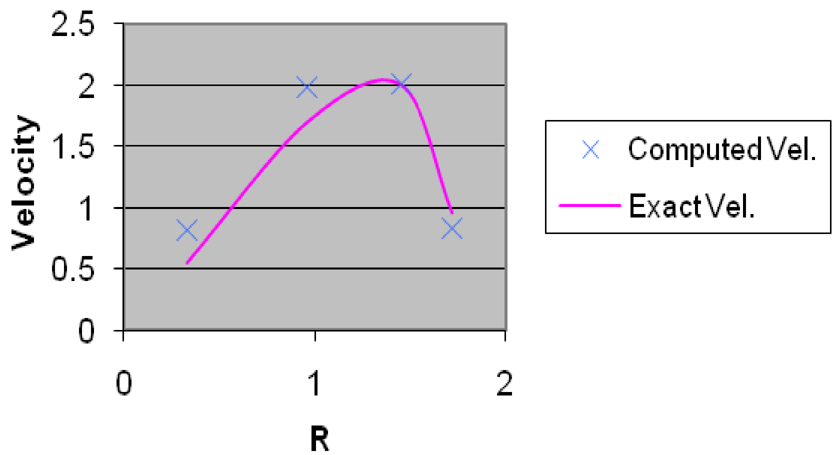


Figure 2: Comparison of computed and exact velocity distribution over the boundary of symmetric aerofoil using for 8 constant panels.

Table (2): Comparison of computed and exact velocity distribution over the boundary of symmetric aerofoil using for 16 constant panels.

ELEMENT	X	Y	R	VELOCITY	EXACT VELOCITY
1	-1.84	.19	1.85	.39927E+00	.53942E+00
2	-1.70	.53	1.78	.11366E+01	.13424E+01
3	-1.43	.80	1.64	.16997E+01	.18696E+01
4	-1.08	.94	1.43	.20016E+01	.20347E+01
5	-.71	.93	1.17	.19946E+01	.18738E+01
6	-.36	.79	.86	.16752E+01	.14667E+01
7	-.09	.51	.52	.10740E+01	.88980E+00
8	.07	.17	.18	.40364E+00	.27208E+00
9	.07	-.17	.18	.40364E+00	.27208E+00
10	-.09	-.51	.52	.10740E+01	.88980E+00
11	-.36	-.79	.86	.16752E+01	.14667E+01
12	-.71	-.93	1.17	.19946E+01	.18738E+01
13	-1.08	-.94	1.43	.20016E+01	.20347E+01
14	-1.43	-.80	1.64	.16997E+01	.18696E+01
15	-1.70	-.53	1.78	.11366E+01	.13424E+01
16	-1.84	-.19	1.85	.39927E+00	.53942E+00

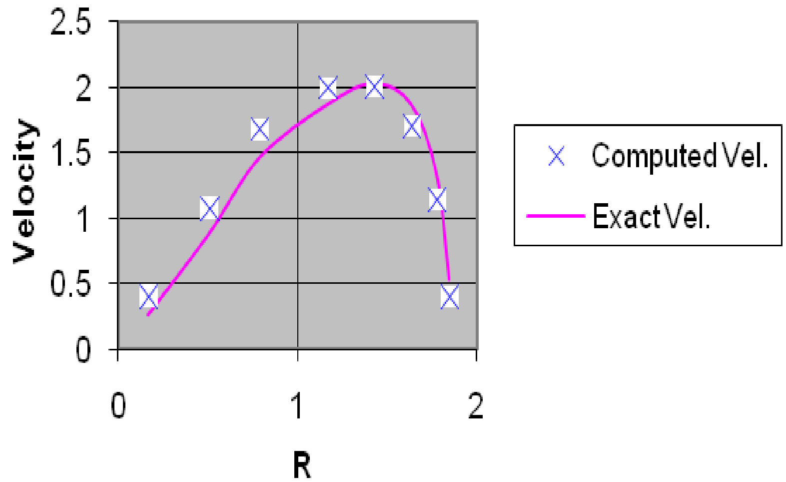


Figure 3: Comparison of computed and exact velocity distribution over the boundary of symmetric aerofoil using for 16 constant panels.

Table (3): Comparison of computed and exact velocity distribution over the boundary of symmetric aerofoil using for 32 constant panels.

ELEMENT	X	Y	R	VELOCITY	EXACT VELOCITY
1	-1.88	.10	1.88	.19768E+00	.34475E+00
2	-1.84	.29	1.87	.58534E+00	.75183E+00
3	-1.77	.47	1.83	.95041E+00	.11596E+01
4	-1.66	.63	1.78	.12787E+01	.15066E+01
5	-1.52	.76	1.70	.15574E+01	.17713E+01
6	-1.36	.87	1.62	.17758E+01	.19448E+01
7	-1.18	.94	1.51	.19254E+01	.20266E+01
8	-.99	.98	1.39	.20000E+01	.20222E+01
9	-.80	.98	1.26	.19967E+01	.19404E+01
10	-.61	.94	1.12	.19152E+01	.17918E+01
11	-.43	.86	.96	.17580E+01	.15870E+01
12	-.27	.75	.80	.15304E+01	.13359E+01
13	-.13	.61	.62	.12394E+01	.10467E+01
14	-.02	.44	.44	.89375E+00	.72575E+00
15	.08	.25	.26	.51262E+00	.39000E+00
16	.14	.07	.16	.26472E+00	.41172E+00
17	.14	-.07	.16	.26472E+00	.41172E+00
18	.08	-.25	.26	.51263E+00	.39000E+00
19	-.02	-.44	.44	.89375E+00	.72575E+00
20	-.13	-.61	.62	.12394E+01	.10467E+01
21	-.27	-.75	.80	.15304E+01	.13359E+01
22	-.43	-.86	.96	.17580E+01	.15870E+01
23	-.61	-.94	1.12	.19152E+01	.17918E+01
24	-.80	-.98	1.26	.19967E+01	.19404E+01
25	-.99	-.98	1.39	.20000E+01	.20222E+01
26	-1.18	-.94	1.51	.19254E+01	.20266E+01
27	-1.36	-.87	1.62	.17758E+01	.19448E+01
28	-1.52	-.76	1.70	.15574E+01	.17713E+01
29	-1.66	-.63	1.78	.12787E+01	.15066E+01
30	-1.77	-.47	1.83	.95041E+00	.11596E+01
31	-1.84	-.29	1.87	.58535E+00	.75183E+00
32	-1.88	-.10	1.88	.19767E+00	.34475E+00

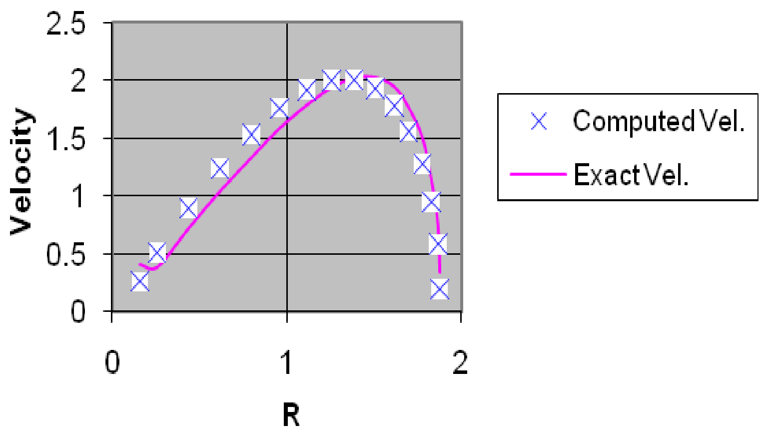


Figure 4: Comparison of computed and exact velocity distribution over the boundary of symmetric aerofoil using for 32 constant panels.

Table (4): Comparison of computed and exact velocity distribution over the boundary of symmetric aerofoil using for 64 constant panels.

ELEMENT	X	Y	R	VELOCITY	EXACT VELOCITY
1	-1.89	.05	1.89	.98563E-01	.27370E+00
2	-1.88	.15	1.89	.29483E+00	.43735E+00
3	-1.86	.24	1.88	.48816E+00	.64522E+00
4	-1.84	.33	1.87	.67685E+00	.85754E+00
5	-1.80	.43	1.85	.85888E+00	.10622E+01
6	-1.75	.51	1.83	.10326E+01	.12532E+01
7	-1.70	.59	1.80	.11964E+01	.14270E+01
8	-1.64	.67	1.77	.13485E+01	.15810E+01
9	-1.57	.74	1.73	.14876E+01	.17135E+01
10	-1.49	.80	1.69	.16121E+01	.18233E+01
11	-1.41	.85	1.65	.17211E+01	.19101E+01
12	-1.32	.90	1.60	.18133E+01	.19737E+01
13	-1.23	.93	1.55	.18879E+01	.20145E+01
14	-1.14	.96	1.49	.19440E+01	.20333E+01
15	-1.04	.98	1.43	.19813E+01	.20309E+01
16	-.94	.99	1.37	.19992E+01	.20086E+01
17	-.85	.99	1.30	.19976E+01	.19676E+01
18	-.75	.98	1.23	.19764E+01	.19091E+01
19	-.65	.96	1.16	.19357E+01	.18346E+01
20	-.56	.93	1.09	.18760E+01	.17455E+01
21	-.47	.89	1.01	.17977E+01	.16429E+01
22	-.38	.84	.93	.17015E+01	.15283E+01
23	-.30	.79	.84	.15881E+01	.14026E+01
24	-.22	.72	.76	.14585E+01	.12669E+01
25	-.15	.65	.67	.13138E+01	.11222E+01
26	-.09	.57	.58	.11552E+01	.96920E+00
27	-.04	.49	.49	.98386E+00	.80860E+00
28	.01	.40	.40	.80145E+00	.64151E+00
29	.06	.30	.31	.61056E+00	.47166E+00
30	.10	.20	.22	.41800E+00	.31801E+00
31	.14	.10	.17	.24484E+00	.29405E+00
32	.19	.02	.19	.97678E+00	.43712E+00
33	.19	-.02	.19	.97678E+00	.43712E+00

34	.14	-.10	.17	.24483E+00	.29405E+00
35	.10	-.20	.22	.41801E+00	.31801E+00
36	.06	-.30	.31	.61056E+00	.47166E+00
37	.01	-.40	.40	.80145E+00	.64151E+00
38	-.04	-.49	.49	.98386E+00	.80860E+00
39	-.09	-.57	.58	.11552E+01	.96920E+00
40	-.15	-.65	.67	.13138E+01	.11222E+01
41	-.22	-.72	.76	.14585E+01	.12669E+01
42	-.30	-.79	.84	.15881E+01	.14026E+01
43	-.38	-.84	.93	.17015E+01	.15283E+01
44	-.47	-.89	1.01	.17977E+01	.16429E+01
45	-.56	-.93	1.09	.18760E+01	.17455E+01
46	-.65	-.96	1.16	.19357E+01	.18346E+01
47	-.75	-.98	1.23	.19764E+01	.19091E+01
48	-.85	-.99	1.30	.19976E+01	.19676E+01
49	-.94	-.99	1.37	.19992E+01	.20086E+01
50	-1.04	-.98	1.43	.19813E+01	.20309E+01
51	-1.14	-.96	1.49	.19440E+01	.20333E+01
52	-1.23	-.93	1.55	.18879E+01	.20145E+01
53	-1.32	-.90	1.60	.18133E+01	.19737E+01
54	-1.41	-.85	1.65	.17211E+01	.19101E+01
55	-1.49	-.80	1.69	.16121E+01	.18233E+01
56	-1.57	-.74	1.73	.14876E+01	.17135E+01
57	-1.64	-.67	1.77	.13485E+01	.15810E+01
58	-1.70	-.59	1.80	.11964E+01	.14270E+01
59	-1.75	-.51	1.83	.10326E+01	.12532E+01
60	-1.80	-.43	1.85	.85890E+00	.10622E+01
61	-1.84	-.33	1.87	.67684E+00	.85754E+00
62	-1.86	-.24	1.88	.48815E+00	.64522E+00
63	-1.88	-.15	1.89	.29486E+00	.43735E+00
64	-1.89	-.05	1.89	.98533E-01	.27370E+00

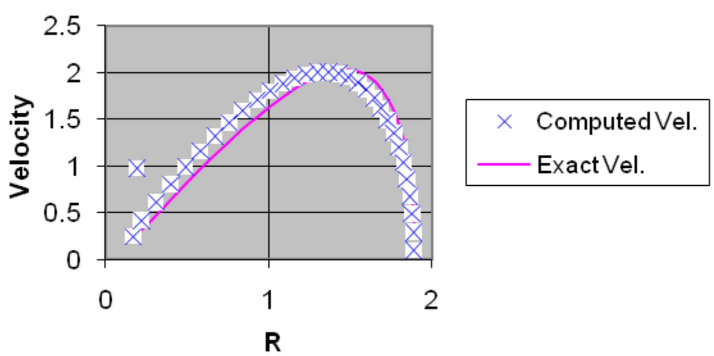


Figure 5: Comparison of computed and exact velocity distribution over the boundary of symmetric aerofoil using for 64 constant panels.

Conclusion: An indirect panel method has been used for the calculation of ideal flow around two – dimensional body. The computed flow velocities obtained using this method is compared with the exact solutions for flows over the surface of a symmetric aerofoil. It is found that from tables (1) to (4) and figures (2) to (5), the results obtained with the indirect panel method for the flow field calculations are excellently close with the exact results for the body under consideration.

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