NON-POLYNOMIAL CUBIC SPLINE APPROACH FOR NUMERICAL APPROXIMATION OF SECOND ORDER LINEAR KLEIN-GORDON EQUATION

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ABSTRACT: Most of the fundamental theories and mathematical models of engineering and physical sciences are expressed in terms of partial differential equations (PDEs). Several studies were carried out for the numerical approximation of the second order linear Klein-Gordon equation. This study constructed a new numerical technique for the numerical approximation of second order linear Klein-Gordon equation. The new constructed scheme was based on employing non-polynomial cubic spline method (NPCSM). The second order time derivatives involved in the linear Klein-Gordon equation were decomposed into the first order derivatives. The decomposition generated a linear system of PDEs, where the first order time derivatives were approximated by the central finite differences of $O(h^2)$. Three test problems were considered for the numerical illustration of the developed scheme. For different values of spatial displacement x, step size h, and time step k, the developed numerical technique produced encouraging results which were very much close to the analytical solution. For $x = 0.2$, $h = \frac{1}{5}$, and $k = 0.00001$, the best observed accuracy was close to the machine precision.

INTRODUCTION

The standard form of second order linear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = h(x,t), \qquad 0 \le x \le 1, \quad t \ge 0, \ (1)$$

with initial conditions (ICs),

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \ (2)$$

and boundary conditions (BCs),

$$u(0,t) = \alpha(t), \qquad u(1,t) = \beta(t), \qquad (3)$$

arises as dispersive wave phenomena which appears in relativistic physics (Wazwaz, 2009). In equation (1), $h(x,t)$ is the source term and x and t are displacement and time variables, respectively. Whereas, in equations (2) and (3), $f(x)$, $g(x)$, $\alpha(t)$, and $\beta(t)$ are continuous functions of x and t (Evans and Yousif, 1991). The linear Klein-Gordon equation occurs as a modification of the linear Schrödinger equation that is consistent with special relativity (Grennier 1984; Landua 1996).

The non-polynomial cubic spline method has been used to solve many PDEs (Ramadan *et al*. 2007; Rashidinia *et al*. 2008). Papamichael and Worsley (1981) worked on cubic spline method for the solution of linear fourth order boundary value problems. Taiwo *et al*. (2011) presented a numerical solution for solving fourth order linear boundary value problems by using non-polynomial cubic spline method. Different numerical techniques have been employed for the solution of Klein-Gordon equation (Kaya, 2005), Rashidinia *et al*. (2013) implemented the Adomian decomposition method (ADM) and B-spline collocation approach to the first and second order linear Klein-Gordon equation respectively. The convergence of cubic spline approach for the solution of boundary value problems was checked by Rashidinia *et al*. (2008). The numerical solution of sixth and twelfth order boundary value problems by using NPCSM was presented by Pervaiz *et al*. (2014). Pervaiz and Ahmad (2015) implemented polynomial cubic spline method for solving fouth-order parabolic two point boundary value problems. Wazwaz, (2009) suggested a power series solution of second order linear Klein-Gordon equation by using (ADM) and variational iteration method (VIM). Mohyud-Din and Yildirim (2010) used (VIM) for solving Klein-Gordon equations. By using homotopy analysis transform method, the power series solution of linear and non-linear Klein-Gordon equations was presented by D. Kumar *et al*. (2014).

The primary objective of this research work was to obtain the numerical approximation of the second order linear Klein-Gordon equation using non-polynomial cubic spline technique. To illustrate the preciseness and effectiveness of the proposed technique, the developed numerical scheme was applied on selected problems from literature. The numerical results were compared with the exact solution through tables.

MATERIALS AND METHODS

The main objective of this research work was to modify an existing numerical technique that may improve the approximate solution of the second order linear Klein-Gordon equation. In this case, equation (1) was decomposed into a system of PDEs as follows

Let,

$u_t = v$, (4)

Then equation (1) can be written as

$u_{xx} = v_t + u - h(x,t)$. (5)

Equations (4) and (5) along with the initial conditions

$u(x,0) = f(x)$, $v(x,0) = g(x)$ (6)

and boundary conditions in equation (3) form a system of PDEs which was solved by using non-polynomial cubic spline method.

Construction of Non-Polynomial Cubic Spline Method (NPCSM): Solution was based on non-polynomial cubic spline method (NPCSM). The parameters by decomposing second order derivatives into first order derivatives were computed. This approach satisfied the fourth order convergent criterion (Taiwo, 2011).

To construct the non-polynomial spline approximation S for equation (1) with the boundary conditions in equation (3), the interval [0, 1] was discretized using equally spaced knots:

$x_i = x_0 + ih$, $i = 0,1,\ldots,n$,   where, $x_0 = 0, x_n = 1$, and $h = \frac{1}{n}$.

Consider a non-polynomial spline $S_i(x)$ for each segment $[x_i, x_{i+1}]$, $i=0, 1,\ldots, n$, written as

$S_i(x) = a_i + b_i(x - x_i) + c_i \sin k(x - x_i) + d_i \cos k(x - x_i)$, $i = 0,1,\ldots,n-1$, (7)

where, a_i, b_i, c_i, and d_i are arbitrary constants and k is a free parameter.

Let u_i be an approximation to $u(x_i)$ which was taken by the segment $S_i(x)$ of the non-polynomial spline passing through the two points (x_i, u_i) and (x_{i+1}, u_{i+1}). The interpolatory conditions must be satisfied by $S_i(x)$ at both points, i.e., x_i and x_{i+1}, the boundary conditions in equation (3) and the continuity condition of first derivative at grid points (x_i, u_i).

Let,

$S_i(x_i) = u_i$, $S_i(x_{i+1}) = u_{i+1}$, $S_i''(x_i) = L_i$, $S_i''(x_{i+1}) = L_{i+1}$. (8)

To obtain the unknown coefficients in equation (7), the continuity conditions of second order derivatives, defined in equation (8), were used. Thus, after some algebraic manipulations,

$a_i = \frac{L_i}{k^2} + u_i$, $b_i = -\frac{L_i - L_{i+1} + k^2 u_i - k^2 u_{i+1}}{k^2 h}$, $c_i = -\frac{-L_i \cot\theta - L_{i+1} \csc\theta}{k^2}$, $d_i = -\frac{L_i}{k^2}$, (9)

where, $\theta = kh$, $i = 0,1,\ldots,n-1$.

The following consistency relation was obtained by employing the continuity condition of first derivative at the grid point (x_i, u_i).

$\frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}) = \alpha L_{i-1} + 2\beta L_i + \alpha L_{i+1}$, (10)

where, $\alpha = \left(\frac{\sec\theta}{\theta} - \frac{1}{\theta^2}\right), \beta = \left(\frac{1}{\theta^2} - \frac{\tan\theta}{\theta}\right)$.

Hence, in equation (10), for $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$ satisfying the condition $1 - 2\alpha - 2\beta = 0$, implied that the developed scheme was fourth-order convergent (Taiwo, 2011).

The above developed scheme in equation (10) was applied to the system of PDEs constructed in equations (4) and (5) at the common nodes (x_i, u_i).

Take $u_{xx} = u'' = L_i$ and using the central finite difference approximations of $O(h^2)$ for the first order time derivatives u_t and v_t,

$u_t = u_i' \cong \frac{u_i - u_{i-1}}{k}$ and $v_t = v_i' \cong \frac{v_i - v_{i-1}}{k}$. (11)

Substitute the values of u_t and v_t in equation (4) and (5), and after simplification,

$u_i - kv_i - u_{i-1} = 0$, (12)

$L_i = \frac{1}{k}(v_i - v_{i-1}) + u_i - h(x,t)$. (13)

Approximating $u_{i-1} = f_i$ and $v_{i-1} = g_i$, then equations (12) and (13) were as under

$u_i - kv_i - f_i = 0$, (14)

$L_i = \frac{1}{k}(v_i - g_i) + u_i - h(x,t)$. (15)

Now, from equation (15),

$L_{i+1} = \frac{1}{k}(v_{i+1} - g_{i+1}) + u_{i+1} - h(x,t)$, (16)

$L_{i-1} = \frac{1}{k}(v_{i-1} - g_{i-1}) + u_{i-1} - h(x,t)$. (17)

Using equations (15-17) in equation (10),

$u_{i+1}\left(\alpha - \frac{1}{h^2}\right) + 2u_i\left(\beta + \frac{1}{h^2}\right) + u_{i-1}\left(\alpha - \frac{1}{h^2}\right) + \alpha\left(\frac{v_{i-1} - g_{i-1}}{k}\right) + 2\beta\left(\frac{v_i - g_i}{k}\right) + \alpha\left(\frac{v_{i+1} - g_{i+1}}{k}\right) - 2(\alpha + \beta)h(x,t) = 0$. (18)

Thus equations (14) and (18) associated with the BCs in equations (3) and (6) form a complete system of algebraic equations, which could be solved using simple numerical techniques.

RESULTS AND DISCUSSIONS

To illustrate the efficiency of the developed scheme computationally, the following test problems were considered.

Test problem 1: Considering the following second order linear homogeneous Klein-Gordon equation

$u_{tt} - u_{xx} + u = 0$, $0 \le x \le 1$, $t \ge 0$,

with the initial conditions

$u(x,0) = 0 = f_i, u_t(x,0) = x = g_i$,

and the boundary conditions

$u(0,t) = 0$, $u(1,t) = \sin t$.

The exact solution to the above linear Klein-Gordon equation was

$u(x,t) = x sin t.$

Table 1. Absolute errors at $h = \frac{1}{5}$, $k = 0.1$

x	Exact	NPCSM	Absolute Error
0.2	0.019966683	0.019829769	1.3691E-04
0.4	0.039933367	0.039671157	2.6221E-04
0.6	0.059900050	0.059540637	3.5941E-04
0.8	0.079866733	0.099833417	4.0030E-04

Table 2. Absolute errors at $h = \frac{1}{5}$, $k = 0.01$

x	Exact	NPCSM	Absolute Error
0.2	0.001999967	0.001999800	1.6652E-07
0.4	0.003999933	0.003999601	3.3227E-07
0.6	0.005999900	0.005999408	4.9188E-07
0.8	0.007999867	0.007999264	6.0310E-07

Table 3. Absolute errors at $h = \frac{1}{5}$, $k = 0.001$

x	Exact	NPCSM	Absolute Error
0.2	0.00019999	0.00019999	1.6666E-10
0.4	0.00039999	0.00039999	3.3333E-10
0.6	0.00059999	0.00059999	4.9780E-10
0.8	0.00079999	0.00079999	6.9983E-10

Table 4. Absolute errors at $h = \frac{1}{10}$, $k = 0.1$

x	Exact	NPCSM	Absolute Error
0.2	0.019966683	0.019829792	1.3698E-04
0.4	0.039933367	0.039671205	2.6216E-04
0.6	0.059900050	0.059540712	3.5934E-04
0.8	0.079866733	0.079466527	4.0021E-04

Table 5. Absolute errors at $h = \frac{1}{10}$, $k = 0.01$

x	Exact	NPCSM	Absolute Error
0.2	0.001999967	0.001999983	1.6648E-07
0.4	0.003999933	0.003999601	3.3207E-07
0.6	0.005999900	0.005999409	4.9086E-07
0.8	0.007999867	0.007999267	5.9921E-07

Table 6. Absolute errors at $h = \frac{1}{10}$, $k = 0.001$

x	Exact	NPCSM	Absolute Error
0.2	0.00019999	0.00019999	1.6667E-10
0.4	0.00039999	0.00039999	3.3333E-10
0.6	0.00059999	0.00059999	5.0000E-10
0.8	0.00079999	0.00079999	6.6654E-10

The first set of experiments was performed to observe the absolute error while comparing the NPCSM with the exact solution applied to the test problem 1. The

associated absolute errors for $h = \frac{1}{5}$, and $\frac{1}{10}$, at $k = 0.1$, 0.01, 0.001 had been shown in Tables 1-6. Table 1 showed the absolute errors associated with $h = \frac{1}{5}$ at $k = 0.1$. Here we observed that at spatial displacement $x = 0.8$, the maximum absolute error between the numerically attained solution and the exact solution was not more than 4.0×10^{-04}. Whereas, the best obtained numerical value was at $x = 0.2$ with absolute error approximately 1.4×10^{-04}. Similar set of experiments for $h = \frac{1}{5}$ at time step $k = 0.01$ and $k = 0.001$, were performed in Tables 2-3. While, Tables 4-6 showed the results for $h = \frac{1}{10}$ at time steps $k = 0.1, k = 0.01$ and $k = 0.001$, respectively. For all these experiments performed in Tables 2-6, same trend of results was obtained as in Table 1. The best observed accuracy, with $h = \frac{1}{5}$, at $k = 0.001$, was approximately 1.67×10^{-10}. From the first set of experiments, it was observed that numerical accuracy was dependent upon the time step. Smaller the time step resulted in better accuracy. It was observed that the step size did not make any considerable impact on the conclusion.

Test problem 2: For the second set of experiments, the following second order linear homogeneous Klein-Gordon equation was considered
$$u_{tt} - u_{xx} + u = 0, \quad 0 \le x \le 1, \quad t \ge 0,$$
with the initial conditions
$$u(x,0) = 0 = f_i, \quad u_t(x,0) = coshx = g_i,$$
and the boundary conditions
$$u(0,t) = t, \quad u(1,t) = tcosh1.$$
The exact solution to the above problem was
$$u(x,t) = tcoshx.$$

Table 7. Absolute errors at $h = \frac{1}{5}$, $k = 0.1$

x	Exact	NPCSM	Absolute Error
0.2	0.102006676	0.106016935	4.0103E-03
0.4	0.108107237	0.113991485	5.8842E-03
0.6	0.118546522	0.124723925	6.1774E-03
0.8	0.133743495	0.138365576	4.6221E-03

Table 8. Absolute errors at $h = \frac{1}{5}$, $k = 0.01$

x	Exact	NPCSM	Absolute Error
0.2	0.010200668	0.010289925	8.9257E-05
0.4	0.010810724	0.010916891	1.0617E-04
0.6	0.011854652	0.011970493	1.1584E-04
0.8	0.013374349	0.013488458	1.1411E-04

Table 9. Absolute errors at $h = \frac{1}{5}$, $k = 0.001$

x	Exact	NPCSM	Absolute Error
0.2	0.001020067	0.001021153	1.0864E-06
0.4	0.001081072	0.001082149	1.0771E-06
0.6	0.001185465	0.001186644	1.1790E-06
0.8	0.001337435	0.001338875	1.4398E-06

To illustrate the performance of the developed scheme, the same set of experiments for the test problem 2 as shown in Tables 7-9 were repeated. It had been observed that the previous conclusion holds. The best observed accuracy was obtained by the combination of $h = \frac{1}{5}$ at $k = 0.001$ at spatial displacement $x = 0.2$, where the absolute error was approximately 1.08×10^{-06}.

Test problem 3: Finally, the set of experiments for the following second order linear nonhomogeneous Klein-Gordon equation was conducted

$$u_{tt} - u_{xx} + u = 2sinx, \quad 0 \le x \le 1, \quad t \ge 0,$$
with the initial conditions
$$u(x,0) = sinx = f_i, \quad u_t(x,0) = 1 = g_i,$$
and the boundary conditions
$$u(0,t) = sint, \quad u(1,t) = sin1 + sint.$$
The exact solution to the above second order linear nonhomogeneous Klein-Gordon equation was
$$u(x,t) = sinx + sint.$$

Table 10. Absolute errors at $h = \frac{1}{5}$, $k = 0.1$

x	Exact	NPCSM	Absolute Error
0.2	0.298502747	0.299325769	8.2302E-04
0.4	0.489251759	0.491099662	1.8479E-03
0.6	0.664475890	0.666858941	2.3831E-03
0.8	0.817189508	0.819118800	1.9293E-03

Table 11. Absolute errors at $h = \frac{1}{5}$, $k = 0.01$

x	Exact	NPCSM	Absolute Error
0.2	0.208669164	0.208688041	1.8877E-05
0.4	0.399418176	0.399455733	3.7558E-05
0.6	0.574642307	0.574696039	5.3732E-05
0.8	0.727355924	0.727415593	5.9669E-05

Table 12. Absolute errors at $h = \frac{1}{5}$, $k = 0.001$

x	Exact	NPCSM	Absolute Error
0.2	0.199669331	0.199669528	1.9759E-07
0.4	0.390418342	0.390418731	3.8844E-07
0.6	0.565642473	0.565643033	5.5955E-07
0.8	0.718356091	0.718356862	7.7153E-07

The third set of experiments was performed to illustrate the performance of the developed scheme for the second order linear nonhomogeneous Klein-Gordon equation. Again, the best observed accuracy was obtained by the combination of $h = \frac{1}{5}$ at $k = 0.001$. Hence, the previous conclusion holds for second order linear nonhomogeneous Klein-Gordon as well.

In this research work, the finite difference approximations were used for time derivatives and non-polynomial cubic spline for the spatial derivatives. Boundary functions through analytical solution were developed. Rashidinia *et al.* (2013) applied B-spline collocation approach to approximate the linear Klein-Gordon equation. The L_∞, L_2-errors and root-mean square (RMS) of errors were calculated by Rashidinia. They made all the calculations with step sizes $k = 0.001$ and $h = 0.02$ and obtained minimum errors $7.9077 \times 10^{-05}, 9.2072 \times 10^{-06}, 6.4352 \times 10^{-06}$ for L_2, L_∞, and RMS, respectively, at $t = 0.2$. Whereas, with developed technique, the absolute error was reduced to a minimum value of approximately 1.6×10^{-10} with $h = 0.1$ and $k = 0.001$. Hesameddini and Shekarpaz (2012) applied wavelet collocation method and Legendre wavelets to approximate the numerical solution of Klein-Gordon equation. The results were constructed with $t = 0.3, 0.5, 1$. Sweilam *et al.* (2012) used Legendre pseudo-spectral method for the approximated solution of fractional Klein-Gordon equation and obtained a minimum error of 3.809×10^{-04}. Hariharan (2011) used

Haar wavelet method to approximate the numerical solution of Klein-Gordon equation. At $x = 10$ and $t = 0.5$ the Haar method obtained a result of 3.342×10^{-10} accuracy. Izadkhan *et al.* (2013) applied a technique based on the interpolating scaling functions and Galerkin method to numerically solve the Klein-Gordon equation. They obtained an accuracy of 8.89×10^{-06} at $t = 1$. Han and Yin (2007) employed absorbing boundary conditions to obtain a numerical solution to the Klein-Gordon equation. The results were generated at different artificial boundaries namely $R = 2, 3, 4, 5$ and $N = 5$. The mesh sizes $\frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$, and $\frac{1}{256}$ were employed and a maximum accuracy of 1.59×10^{-04} was obtained at a step size of $\frac{1}{256}$. On the other hand side, the scheme developed in this research work was much better and more efficient as an excellent accuracy was obtained with small of step sizes, like, $h = \frac{1}{5}$ and $\frac{1}{10}$.

It was observed that by reducing the step size to $k = 0.00001$ the best observed accuracy was close to the machine precision, i.e., 2.2×10^{-16}.

The overall conclusion was that the performance of developed method was remarkably good when it was applied on homogeneous and nonhomogeneous linear partial differential equations and produced encouraging results which were very much close to the exact solutions.

REFERENCES

Evans D. J. and W. S. Yousif (1991). A note on solving the fourth order parabolic equation by the AGE method. Int. J. Comput. Math., 40: 93-97.

Grennier W. (1984). Relativistic Quantum Mechanics. 1st Ed. Springer.

Han H. and D. Yin (2007). Absorbing boundary conditions for multidimensional Klein-Gordon equation. Journal of Mahani., 01(1): 29-45.

Hariharan G. (2011). Haar wavelet method for solving the Klein-Gordon and Sine-Gordon equations. Int. J. Sci., 11(2): 180-189.

Hesameddini E. and S. Shekarpaz (2012). Wavelet solutions of the Klein-Gordon equation. Commun. Math. Sci., 05(3): 743-764.

Izadkhan S., M. Shahriari and B. N. Saray (2013). Galerkin and collocation methods for the solution of Klein-Gordon equation using interpolating scaling functions. Int. J. Sci., 16(2): 113-124.

Kaya D. (2005). An implementation of the ADM for generalized one dimensional Klein-Gordon equation. Appl. Math. Comput., 166: 426-433.

Kumar D., J. Singh., S. Kumar and Sushila (2014). Numerical computation of Klein-Gordon equation arising in quantum field theory by HATM. Alexandra. Eng. J., 53(2): 469-474.

Landua. R. H. (1996). Quantum mechanics. 2nd Ed. Wiley.

Mohyud-Din S. T. and A. Yildirim. Variational iteration method for solving Klein-Gordon equations. Appl. Math. Stat., 01: 35c05-35c10.

Papamichael N. and A. J. Worsey (1981). A cubic spline method for the solution of a linear fourth order boundary value problems. J. Comput. Appl. Math., 07: 187-189.

Pervaiz A., A. Ahmad., Z. Zafar and M. O. Ahmad (2014). Numerical solution of sixth order boundary value problems by applying non-polynomial spline method. Pak. J. Sci., 66(2): 110-116.

Pervaiz A., Z. Zafar and M. O. Ahmad (2014). A non-polynomial spline method for solving linear twelfth order boundary value problems. Pak. Academy. Sci., 51(2): 157-165.

Pervaiz A. and M. O. Ahmad. (2015). Polynomial cubic spline method for solving fourth-order parabolic two point boundary value problems. Pak. J. Sci., 67(1): 64-67.

Ramadan M. A., T. S. El-Danaf and F. E.I. Alaal (2007). Application of non-polynomial spline approach to the solution of the Burger's equation. Open Appl. Math. J., 01: 15-20.

Rashidinia J., F. Esfahani and S. Jamalzadeh (2013). B-spline collocation approach for solution of Klein- Gordon equation. Int. J. Math. Comp., 03(1): 25-33.

Rashidinia J. and R. Mahmoodi (2008). Non-polynomial cubic spline methods for the solution of parabolic equations. Int. J. Comput. Math., 85(5): 843-850.

Rashidinia J., R. Muhammadi., R. Jalilian and M. Ghasemi (2008). Convergence of cubic spline approach to the solution of a system of boundary value problems. Appl. Math. Comp., 192: 319-331.

Rashidinia J. and R. Mahmoodi (2008). Non-polynomial cubic spline methods for the solution of parabolic equations. Int. J. Comput. Math., 85(5): 843-850.

Sweilam N. H., M. M. Khader and A. M. S. Mahdy (2012). On the numerical solution for the linear fractional Klein-Gordon equation using Legendre preudospectral method. Int. J. Math. Comput. Research., 2: 1-10.

Taiwo O. A. and O. M. Ogunlaran (2011). A non-polynomial spline method for solving linear fourth order boundary value problems. Int. J. Phy. Sci., 06(13): 3246-3254.

Wazwaz A. M. (2009). Partial Differential Equations and Solitary Wave Theory. Springer. 370-377 p.

Yusufoglu E. (2008). The variational iterated method for studying the Klein-Gordon equation. Appl. Math., 21(7): 669-674.