ZZ FOURTH ORDER COMPACT BVM FOR ONE DIMENSIONAL ADVECTION DIFFUSION EQUATION

Z. Zafar, M. T. Hussain, A. Pervaiz[1], M. O. Ahmad[1], M. Kalim[2]

Faculty of Engineering, University of Central Punjab, Lahore, Pakistan.
[1]Department of Mathematics, University of Engineering & Technology, Lahore, Pakistan.
[2]Department of Mathematics, NCBA & E, Lahore, Pakistan.
Corresponding Author E-mail: zainulabadin1@hotmail.com

ABSTRACT: In this paper we combine the boundary value method (for discretizing the temporal variable) and finite difference scheme (for discretizing the spatial variables) to numerically solve the one dimensional Advection Diffusion Equation. We first employ a fourth order compact scheme to discretize the spatial derivatives. Then a linear system of ordinary differential equation is obtained. Then we apply a fourth order scheme of boundary value method to approach this system. After this, we use the central difference scheme for the temporal variables. Therefore, this scheme can achieve fourth order accuracy for both temporal and spatial variables. Numerical applications are performed to check the correctness and effectiveness of this compact difference scheme, compared with finite difference scheme.

Key words: Finite Difference scheme, Compact Method, one dimensional Advection Diffusion Equation, BVM.

INTRODUCTION

The compact method is a new finite difference approximation. We firstly dicretize the spatial variables by a fourth order compact difference scheme. Since the boundary conditions are homogeneous, the resulting system after discretization for the spatial direction is an initial value problem. After this, we use the central difference scheme for the temporal variables.

Compact method: Let us take the one dimensional Advection Diffusion Equation as

$$\frac{\partial u(x,t)}{\partial t} + \beta \frac{\partial u(x,t)}{\partial x} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} \quad (1)$$

with the following initial condition

$$u(x,0) = f(x) \; x \in \Omega \quad (2)$$

and with the homogeneous Dirichlet boundary conditions

$$u(0,t) = 0 \quad (3)$$

$$u(l,t) = 0 \quad (4)$$

$$for \quad 0 \le x \le l, \quad t > 0$$

where $\Omega \epsilon [0,l]$, $'f'$ is sufficiently smooth and its higher derivatives exist. Where α, β are non negative constant.

Let $h = \frac{l}{N}$, be the uniform spatial mesh width. The spatial domain Ω can be subdivided by

$$x_i = i\,\Delta x = ih \quad \text{for } i = 0,1,2,3,\dots l$$

We also discretize the temporal variable $'t'$ by $t_n = nk$, $n = 0,1,2,\dots,T$, where k is the step size in the temporal direction, T is the number of the time steps and the final time is $T_f = T\,k$.

We know by the Taylor Series

$$u_i^{n+1} = u_i^n + k\,u_t|_i^n + \frac{k^2}{2}u_{tt}|_i^n + \frac{k^3}{6}u_{ttt}|_i^n + \frac{k^4}{24}u_{tttt}|_i^n + \frac{k^5}{120}u_{ttttt}|_i^n + \cdots \quad (5)$$

$$u_i^{n-1} = u_i^n - k\,u_t|_i^n + \frac{k^2}{2}u_{tt}|_i^n - \frac{k^3}{6}u_{ttt}|_i^n + \frac{k^4}{24}u_{tttt}|_i^n - \frac{k^5}{120}u_{ttttt}|_i^n + \cdots \quad (6)$$

Subtracting equation (5) from (6) and dividing by $2k$.

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\,k} = u_t|_i^n + O(k^2)$$

Or

$$\frac{1}{2\,k}\delta_t u_i^n = u_t|_i^n + O(k^2) \quad (7)$$

where $\delta_t u_i^n = u_i^{n+1} - u_i^{n-1}$

Similarly

$$\frac{1}{2\,h}\delta_x u_i^n = u_x|_i^n + O(h^2) \quad (8)$$

where $\delta_x u_i^n = u_{i+1}^n - u_{i-1}^n$

Also we have

$$\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} = u_{xx}|_i^n + \frac{h^2}{12}u_{xxxx}|_i^n + O(h^4)$$

$$\frac{1}{h^2}\delta_x^2 u_i^n = u_{xx}|_i^n + \frac{h^2}{12}u_{xxxx}|_i^n + O(h^4) \quad (9)$$

where $\delta_x^2 u_i^n = u_{i+1}^n - 2u_i^n + u_{i-1}^n$.

Replace u_i^n by $u_t|_i^n$ in equation (8) and (9).

$$\frac{1}{2h}\delta_x u_t|_i^n = u_{xt}|_i^n + O(h^2) \quad (10)$$

$$\frac{1}{h^2}\delta_x^2 u_t|_i^n = u_{xxt}|_i^n + \frac{h^2}{12}u_{xxxxt}|_i^n + O(h^4) \quad (11)$$

By equation (1)

$$\frac{\partial^2}{\partial x^2}[u_t]_i^n = \alpha \frac{\partial^4 u}{\partial x^4}| - \beta \frac{\partial^3 u}{\partial x^3}$$

Or $$\left.\frac{\partial^4 u}{\partial x^4}\right|_i^n = \frac{1}{\alpha}\frac{\partial^2}{\partial x^2}[u_t]_i^n + \frac{\beta}{\alpha}\left.\frac{\partial^3 u}{\partial x^3}\right|_i^n \quad (12)$$

But by equation (11)

$$\frac{1}{h^2}\delta_x^2 u_t|_i^n = u_{xxt}|_i^n + O(h^2) \quad (13)$$

Therefore equation (12) becomes

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_i^n = \frac{1}{\alpha}\left[\frac{1}{h^2}\delta_x^2 u_t|_i^n - O(h^2)\right] + \frac{\beta}{\alpha}\left.\frac{\partial^3 u}{\partial x^3}\right|_i^n$$

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_i^n = \frac{1}{\alpha h^2}\delta_x^2 u_t|_i^n + \frac{\beta}{\alpha}\left.\frac{\partial^3 u}{\partial x^3}\right|_i^n + O(h^2) \quad (14)$$

Also by (1)

$$\frac{\partial}{\partial x}[u_t]_i^n = \alpha\left.\frac{\partial^3 u}{\partial x^3}\right|_i^n - \beta\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n$$

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_i^n = \frac{1}{\alpha}\frac{\partial}{\partial x}[u_t]_i^n + \frac{\beta}{\alpha}\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n \quad (15)$$

Using equation (15) in equation (14), we have

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_i^n = \frac{1}{\alpha h^2}\delta_x^2 u_t|_i^n + \frac{\beta}{\alpha}\left(\frac{1}{\alpha}\frac{\partial}{\partial x}[u_t]_i^n + \frac{\beta}{\alpha}\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n\right) + O(h^2)$$

(16)

Substitute equation (16) in equation (9), we have

$$\frac{1}{h^2}\delta_x^2 u_i^n = u_{xx}|_i^n + \frac{h^2}{12}\left[\frac{1}{\alpha h^2}\delta_x^2 u_t|_i^n + \frac{\beta}{\alpha}\left(\frac{1}{\alpha}\frac{\partial}{\partial x}[u_t]_i^n + \frac{\beta}{\alpha}\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n\right)\right] + O(h^4)$$

$$\frac{1}{h^2}\delta_x^2 u_i^n = u_{xx}|_i^n + \frac{1}{12\alpha}\delta_x^2 u_t|_i^n + \frac{\beta h^2}{12\alpha^2}\frac{\partial}{\partial x}[u_t]_i^n + \frac{h^2\beta^2}{12\alpha^2}\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n + O(h^4)$$

$$\frac{1}{h^2}\delta_x^2 u_i^n = \left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)u_{xx}|_i^n + \frac{1}{12\alpha}\delta_x^2 u_t|_i^n + \frac{\beta h^2}{12\alpha^2}\frac{\partial}{\partial x}[u_t]_i^n + O(h^4)$$

(17)

By equation (1) we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha}\frac{\delta u}{\partial t} + \frac{\beta}{\alpha}\frac{\delta u}{\partial x} \quad (18)$$

But

$$u_x|_i^n = \frac{1}{2h}\delta_x u_i^n - O(h^2)$$

So equation (18) becomes

$$u_{xx}|_i^n = \frac{1}{\alpha}u_t|_i^n + \frac{\beta}{\alpha}\left(\frac{1}{2h}\delta_x u_i^n - O(h^2)\right)$$

$$u_{xx}|_i^n = \frac{1}{\alpha}u_t|_i^n + \frac{\beta}{2h\alpha}\delta_x u_i^n - O(h^2) \quad (19)$$

Also

$$u_{xt}|_i^n = \frac{1}{2h}\delta_x u_t|_i^n - O(h^2) \quad (20)$$

Substituting equations (19) and (20), in eq. (17) we have

$$\frac{1}{h^2}\delta_x^2 u_i^n = \left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\left(\frac{1}{\alpha}u_t|_i^n + \frac{\beta}{2h\alpha}\delta_x u_i^n - O(h^2)\right) + \frac{1}{12\alpha}\delta_x^2 u_t|_i^n + \frac{\beta h^2}{12\alpha^2}\left(\frac{1}{2h}\delta_x u_t|_i^n - O(h^2)\right) + O(h^4)$$

$$\frac{1}{h^2}\delta_x^2 u_i^n = \frac{1}{\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)u_t|_i^n + \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\delta_x u_i^n + \frac{1}{12\alpha}\delta_x^2 u_t|_i^n + \frac{\beta h}{24\alpha^2}\delta_x u_t|_i^n + O(h^4)$$

$$\frac{1}{h^2}\delta_x^2 u_i^n - \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\delta_x u_i^n = \frac{1}{\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)u_t|_i^n + \frac{1}{12\alpha}\delta_x^2 u_t|_i^n + \frac{\beta h}{24\alpha^2}\delta_x u_t|_i^n + O(h^4)$$

After ignoring the truncation error

$$\frac{1}{h^2}\delta_x^2 u_i^n - \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\delta_x u_i^n = \frac{1}{\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)u_t|_i^n + \frac{1}{12\alpha}\delta_x^2 u_t|_i^n + \frac{\beta h}{24\alpha^2}\delta_x u_t|_i^n$$

(21)

Let us take

$$u_t|_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2k} - O(k^2)$$

Therefore equation (21) becomes

$$\frac{1}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) - \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)(u_{i+1}^n - u_{i-1}^n) = \frac{1}{\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\left(\frac{u_i^{n+1} - u_i^{n-1}}{2k} - O(k^2)\right) + \frac{1}{12\alpha}\delta_x^2\left(\frac{u_i^{n+1} - u_i^{n-1}}{2k} - O(k^2)\right) + \frac{\beta h}{24\alpha^2}\delta_x\left(\frac{u_i^{n+1} - u_i^{n-1}}{2k} - O(k^2)\right)$$

After ignoring the truncation error

$$\frac{1}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) - \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)(u_{i+1}^n - u_{i-1}^n) = \frac{1}{\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\left(\frac{u_i^{n+1} - u_i^{n-1}}{2k}\right) + \frac{1}{12\alpha}\delta_x^2\left(\frac{u_i^{n+1} - u_i^{n-1}}{2k}\right) + \frac{\beta h}{24\alpha^2}\delta_x\left(\frac{u_i^{n+1} - u_i^{n-1}}{2k}\right)$$

$$\frac{1}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) - \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)(u_{i+1}^n - u_{i-1}^n) = \frac{1}{2k\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)(u_i^{n+1} - u_i^{n-1}) + \frac{1}{24\alpha k}\delta_x^2(u_i^{n+1} - u_i^{n-1}) + \frac{\beta h}{48\alpha^2 k}\delta_x(u_i^{n+1} - u_i^{n-1})$$

$$\frac{1}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) - \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)(u_{i+1}^n - u_{i-1}^n) = \frac{1}{2k\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)(u_i^{n+1} - u_i^{n-1}) + \frac{1}{24\alpha k}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} - u_{i+1}^{n-1} + 2u_i^{n-1} - u_{i-1}^{n-1}) + \frac{\beta h}{48\alpha^2 k}(u_{i+1}^{n+1} - u_{i-1}^{n+1} - u_{i+1}^{n-1} + u_{i-1}^{n-1})$$

$$\left(\frac{1}{h^2} - \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\right)u_{i+1}^n + \left(\frac{1}{h^2} + \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\right)u_{i-1}^n - \frac{2}{h^2}u_i^n = \frac{1}{2k\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2} - \frac{1}{6}\right)u_i^{n+1} - \frac{1}{2k\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2} - \frac{1}{6}\right)u_i^{n-1} + \left(\frac{1}{24\alpha k} + \frac{\beta h}{48\alpha^2 k}\right)u_{i+1}^{n+1} + \left(\frac{1}{24\alpha k} - \frac{\beta h}{48\alpha^2 k}\right)u_{i-1}^{n+1} - \left(\frac{1}{24\alpha k} + \frac{\beta h}{48\alpha^2 k}\right)u_{i+1}^{n-1} - \left(\frac{1}{24\alpha k} - \frac{\beta h}{48\alpha^2 k}\right)u_{i-1}^{n-1}$$

$$\frac{1}{24\alpha k}\left(1 - \frac{\beta h}{2\alpha}\right)u_{i-1}^{n+1} + \frac{1}{2k\alpha}\left(\frac{5}{6} + \frac{h^2\beta^2}{12\alpha^2}\right)u_i^{n+1} + \frac{1}{24\alpha k}\left(1 + \frac{\beta h}{2\alpha}\right)u_{i+1}^{n+1} = \left(\frac{1}{h^2} - \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\right)u_{i+1}^n - \frac{2}{h^2}u_i^n + \left(\frac{1}{h^2} + \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\right)u_{i-1}^n + \frac{1}{24\alpha k}\left(1 - \frac{\beta h}{2\alpha}\right)u_{i-1}^{n-1} + \frac{1}{2k\alpha}\left(\frac{5}{6} + \frac{h^2\beta^2}{12\alpha^2}\right)u_i^{n-1} + \frac{1}{24\alpha k}\left(1 + \frac{\beta h}{2\alpha}\right)u_{i+1}^{n-1}$$

$$\Pi\, u_{i-1}^{n+1} + \Sigma u_i^{n+1} + \Upsilon\, u_{i+1}^{n+1} = \Phi\, u_{i+1}^n + \Psi\, u_i^n + \Omega\, u_{i-1}^n + \Pi\, u_{i-1}^{n-1} + \Sigma u_i^{n-1} + \Upsilon\, u_{i+1}^{n-1}$$

(22)

where

$$\frac{1}{24\alpha k}\left(1 - \frac{\beta h}{2\alpha}\right) = \Pi, \qquad \frac{1}{2k\alpha}\left(\frac{5}{6} + \frac{h^2\beta^2}{12\alpha^2}\right) = \Sigma,$$

$$\frac{1}{24\alpha k}\left(1 + \frac{\beta h}{2\alpha}\right) = \Upsilon,$$

$$\left(\frac{1}{h^2} - \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\right) = \Phi, \qquad -\frac{2}{h^2} = \Psi,$$

$$\left(\frac{1}{h^2} + \frac{\beta}{2h\alpha}\left(1 + \frac{h^2\beta^2}{12\alpha^2}\right)\right) = \Omega$$

Equation (22) can be written in matrix form for $i = 1, 2, \ldots, l-1$.

$$\begin{bmatrix} \Sigma & \Upsilon & 0 & & 0 \\ \Pi & \Sigma & \Upsilon & & \\ 0 & \Pi & \ddots & \ddots & 0 \\ & & \ddots & \Sigma & \Upsilon \\ 0 & & 0 & \Pi & \Sigma \end{bmatrix}\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{l-1}^{n+1} \end{bmatrix} = \begin{bmatrix} \Psi & \Phi & 0 & & 0 \\ \Omega & \Psi & \Phi & & \\ 0 & \Omega & \ddots & \ddots & 0 \\ & & \ddots & \Psi & \Phi \\ 0 & & 0 & \Omega & \Psi \end{bmatrix}\begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_{l-1}^n \end{bmatrix} +$$

$$\begin{bmatrix} \Sigma & \Upsilon & 0 & & 0 \\ \Pi & \Sigma & \Upsilon & & \\ 0 & \Pi & \ddots & \ddots & 0 \\ & & \ddots & \Sigma & \Upsilon \\ 0 & & 0 & \Pi & \Sigma \end{bmatrix} \begin{bmatrix} u_1^{n-1} \\ u_2^{n-1} \\ \vdots \\ \vdots \\ u_{l-1}^{n-1} \end{bmatrix} \quad (23)$$

Observing that the implicit scheme is three level in time. Since u_i^0 is given, we need to evaluate u_i^1 for the next time level which can be evaluated as

$$\frac{u_i^{n+1} - u_i^n}{k} = u_t|_i^n + \frac{k}{2} u_{tt}|_i^n + O(k^2)$$

If we replace $n = 0$, then the above equation gives

$$\frac{u_i^1 - u_i^0}{k} = u_t|_i^0 + \frac{k}{2} u_{tt}|_i^0 + O(k^2)$$

Or

$$\frac{u_i^1 - u_i^0}{k} = u_t|_i^0 + O(k)$$

$$u_i^1 - u_i^0 = k\, u_t|_i^0 + O(k^2)$$

$$u_i^1 = u_i^0 + k\, u_t|_i^0 + O(k^2)$$

But by equation (1) we have,

$$\frac{\delta u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\delta u}{\partial x}$$

$$u_i^1 = u_i^0 + k \left[\alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\delta u}{\partial x}\right]\Big|_i^0 + O(k^2)$$

$$u_i^1 = u_i^0 + k\alpha u_{xx}|_i^0 - k\beta u_x|_i^0 + O(k^2)$$

But $f''(x_i) = u_{xx}|_i^0 = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$,

$f'(x_i) = u_x|_i^0 = \frac{f(x_{i+1}) - f(x_{i-1})}{2\,h}$ and

$u_i^0 = f(x_i)$

$$u_i^1 = f(x_i) + k\alpha\left[\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}\right] - k\beta\left[\frac{f(x_{i+1}) - f(x_{i-1})}{2\,h}\right] + O(k^2)$$

$$u_i^1 = f(x_i) + \left(\frac{k\alpha}{h^2} - \frac{k\beta}{2h}\right) f(x_{i+1}) + \left(\frac{k\alpha}{h^2} + \frac{k\beta}{2h}\right) f(x_{i-1}) - \frac{2k\alpha}{h^2} f(x_i) + O(k^2)$$

$$u_i^1 = \left(1 - \frac{2k\alpha}{h^2}\right) f(x_i) + \left(\frac{k\alpha}{h^2} - \frac{k\beta}{2h}\right) f(x_{i+1}) + \left(\frac{k\alpha}{h^2} + \frac{k\beta}{2h}\right) f(x_{i-1}) + O(k^2)$$

So after neglecting the truncation error

$$u_i^1 = \left(1 - \frac{2k\alpha}{h^2}\right) f(x_i) + \left(\frac{k\alpha}{h^2} - \frac{k\beta}{2h}\right) f(x_{i+1}) + \left(\frac{k\alpha}{h^2} + \frac{k\beta}{2h}\right) f(x_{i-1})$$

(24)

for each $i = 1,2,\ldots(l-1)$.

Finite Difference Scheme: To set up the finite difference scheme for eq. (1), select an integer l and the values of t from 0 to ∞ then the mesh points (x_i, t_n) are

$x_i = i\,\Delta x = ih$ for $i = 0,1,2,3,\ldots l$

$t_n = n\,\Delta t = nk$ for $n = 0,1,2,3,\ldots$

At any interior mesh points (x_i, t_n), then the Advection Diffusion Equation (1) becomes

$$\frac{\partial u(x_i,t_n)}{\partial t} + \beta \frac{\partial u(x_i,t_n)}{\partial x} = \alpha \frac{\partial^2 u(x_i,t_n)}{\partial x^2} \quad (25)$$

The method is obtained using the central difference approximation for the first and second order partial derivatives.

So that eq. (25) becomes

$$\frac{1}{2(\Delta t)}(u_i^{n+1} - u_i^{n-1}) - \frac{(\Delta t)^2}{6}\frac{\partial^3 u(x_i,\mu_n)}{\partial t^3} + \frac{\beta}{2(\Delta x)}(u_{i+1}^n - u_{i-1}^n) - \frac{(\Delta x)^2}{6}\frac{\partial^3 u(\xi_i,t_n)}{\partial x^3} = \frac{\alpha}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) - \frac{\alpha(\Delta x)^2}{12}\frac{\partial^4 u(\xi_i,t_n)}{\partial x^4}$$

Where $\xi_i = (x_i, x_{i+1})$ and $\mu_n = (t^n, t^{n+1})$

Neglecting the truncation error leads to the difference equation.

$$\frac{1}{2(\Delta t)}(u_i^{n+1} - u_i^{n-1}) + \frac{\beta}{2(\Delta x)}(u_{i+1}^n - u_{i-1}^n) = \frac{\alpha}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$(u_i^{n+1} - u_i^{n-1}) = \frac{2\alpha(\Delta t)}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) - \frac{\beta(\Delta t)}{(\Delta x)}(u_{i+1}^n - u_{i-1}^n)$$

$$u_i^{n+1} = \left(\frac{2\alpha(\Delta t)}{(\Delta x)^2} - \frac{\beta(\Delta t)}{(\Delta x)}\right)u_{i+1}^n - \frac{4\alpha(\Delta t)}{(\Delta x)^2}u_i^n + \left(\frac{2\alpha(\Delta t)}{(\Delta x)^2} + \frac{\beta(\Delta t)}{(\Delta x)}\right)u_{i-1}^n + u_i^{n-1}$$

Letting $\left(\frac{2\alpha(\Delta t)}{(\Delta x)^2} - \frac{\beta(\Delta t)}{(\Delta x)}\right) = \Lambda$, $-\frac{4\alpha(\Delta t)}{(\Delta x)^2} = \Pi$, $\left(\frac{2\alpha(\Delta t)}{(\Delta x)^2} + \frac{\beta(\Delta t)}{(\Delta x)}\right) = \Psi$

So $u_i^{n+1} = \Lambda\, u_{i+1}^n + \Pi u_i^n + \Psi\, u_{i-1}^n + u_i^{n-1}$

(26)

This equation holds for each $i = 1,2,\ldots,(l-1)$. The boundary conditions give

$$u_0^n = u_l^n = 0 \quad (27)$$

for each $n = 1,2,\ldots$.

And the initial condition implies that

$$u_i^0 = f(x_i) \quad (28)$$

for $i = 1,2,\ldots,(l-1)$.

Writing in matrix form for $i = 1,2,\ldots,(l-1)$, we have

$$\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{l-1}^{n+1} \end{bmatrix} = \Lambda \begin{bmatrix} \Pi & \Lambda & 0 & & 0 \\ \Psi & \Pi & \Lambda & & \\ 0 & \Psi & \ddots & \ddots & 0 \\ & & \ddots & \Pi & \Lambda \\ 0 & & 0 & \Psi & \Pi \end{bmatrix} \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_{l-1}^n \end{bmatrix} + \begin{bmatrix} u_1^{n-1} \\ u_2^{n-1} \\ \vdots \\ \vdots \\ u_{l-1}^{n-1} \end{bmatrix}$$

(29)

Equations (26) and (27) imply that the $(n+1)^{th}$ time steps requires values from the $(n)^{th}$ and $(n-1)^{th}$ time steps. This produces a minor starting problem since values of $n = 1$ which is needed, in equation (26) to compute u_i^2 must be obtained from the initial value condition.

$u_t|_i^0 = f(x_i), \qquad 0 \le x \le l$

A better approximation $u_t|_i^0$ can be obtained rather easily, particularly when the second derivative of $'f'$ at

$'x_i'$ can be determined and it is already obtained in equation (24).

Test Problem: Let us consider the Advection Diffusion equation as

$$\frac{\partial u(x,t)}{\partial t} + 0.1\frac{\partial u(x,t)}{\partial x} = 0.2\frac{\partial^2 u(x,t)}{\partial x^2}$$

in the interval $0 < x < 1$. The boundary conditions are

$$u(0,t) = u(1,t) = 0$$

and the initial conditions are

$$u(x,0) = e^{0.25\,x}\sin\pi x, \quad 0 \le x \le 1, \quad t > 0.$$

The Exact Solution is

$$u(x,t) = e^{[0.25x-(0.0125+0.2\pi^2)t]}\sin\pi x.$$

Comparison of the Numerical Results at $t = 0.02$

Table 1: Finite Difference Method

x_i	FDM	Exact	Error
0.000000000	0.000000000	0.000000000	0.000000000
0.100000000	0.303654188	0.304499030	0.000844842
0.200000000	0.590362070	0.593853831	0.003491761
0.300000000	0.830537995	0.838061512	0.007523517
0.400000000	0.997952755	1.010140896	0.012188141
0.500000000	1.072519616	1.089012742	0.016493126
0.600000000	1.042588651	1.061931849	0.019343198
0.700000000	0.906495419	0.926201224	0.019705805
0.800000000	0.673174928	0.689959764	0.016784836
0.900000000	0.361737330	0.371915936	0.010178606
1.000000000	0.000000000	0.000000000	0.000000000

Table 2: Fourth Order compact Method

x_i	FOCM	Exact	Error
0.000000000	0.000000000	0.000000000	0.000000000
0.100000000	0.304614116	0.304499030	0.000115086
0.200000000	0.593954886	0.593853831	0.000101055
0.300000000	0.838149715	0.838061512	0.000088203
0.400000000	1.010205466	1.010140896	0.000064570
0.500000000	1.089046165	1.089012742	0.000033423
0.600000000	1.061929093	1.061931849	0.000002756
0.700000000	0.926160487	0.926201224	0.000040767
0.800000000	0.689884898	0.689959764	0.000074866
0.900000000	0.371796910	0.371915936	0.000119026
1.000000000	0.000000000	0.000000000	0.000000000

For graph see Figure 1

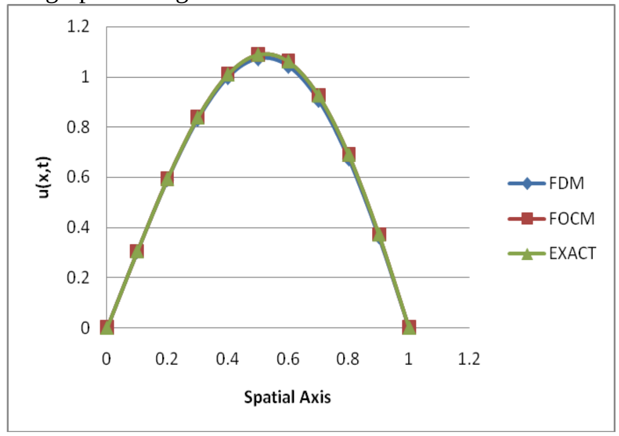

FIGURE 1: Comparison of FDM, FOCM and EXACT Solutions

Conclusion: In this paper, numerical solutions of the one-dimensional Advection Diffusion Equation are derived using Finite Difference Method (FDM) and ZZ Fourth Order Compact Method (FOCM). ZZ Fourth Order Compact Method is known to be a powerful device for solving functional equations. From the solutions of advection diffusion equation, we note that this method, gives better results than the usual second order method.

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