IMPROVEMENTS OF THE GENERALIZED REVERSE AM-GM OPERATOR INEQUALITY FOR UNITAL POSITIVE LINEAR MAPS

S. Karim[1], I. Ali[2] and M. Mushtaq

Department of Basic Sciences and Humanities U.E.T Lahore FSD Campus

ABSTRACT: In this article, we shall give generalized forms of the operator reverse Arithmetic mean -Geometric mean inequality for unital positive linear maps on Hilbert Space operators. Our obtained results are sharper as compare to some recent published results. We shall present more appropriate inequalities.

Index Terms: Positive operators, operator AM-GM inequality, operator reverse AM-GM inequality, operator norm unital positive linear maps.

(Received 00.10.2022 Accepted 17.11.2022)

INRODUCTION

We reserve m_1, m_2, M_1 and M_2 for scalars, other capital letters are used to present the general elements of C^* on Hilbert space $(\mathcal{H}, <.,>)$ whose identity is denoted by I. Operator norm is presented by$\|.\|$. An operator $A_1 \in L(\mathcal{H})$ is known as positive operator if$\langle A_1 x\rangle \geq 0\ \forall x \in \mathcal{H}$ and it is known to be strictly positive if $\langle A_1 x\rangle > 0\ \forall x \in \mathcal{H}\setminus\{0\}$, symbolically we can write these operators by $A_1 \geq 0$ and $A_1 > 0$ respectively .If $B_1 - A_1 \geq 0$,It describes that $B_1 \geq A_1$.A linear map ϕ is named as positive if $\phi(A_1) \geq 0$ whenever$A_1 \geq 0$,we named it unital if for identity operator $\phi(I) = I$,if $A_1, B_1 \in L(\mathcal{H})$ be positive invertible operator t-weighted arithmetic and geometric mean can be defined as respectively as

$$A \nabla_t B = tA + (1-t)B$$

$$A \#_t B = A_1^{\frac{1}{2}}\left(A_1^{\frac{-1}{2}} B_1 A_1^{\frac{-1}{2}}\right)^t A^{\frac{1}{2}}$$

Where= $t \in [0,1]$. When $t = \frac{1}{2}$,we simply write

$$A \nabla_t B = tA + (1-t)B,$$

$$A \#_t B = A_1^{\frac{1}{2}}\left(A_1^{\frac{-1}{2}} B_1 A_1^{\frac{-1}{2}}\right)^t A^{\frac{1}{2}}.$$

For $A_1, B_1 \in L(\mathcal{H})$,the well-known AM-GM operator inequality is defined as

$(A_1 \# B_1) \leq \left(\frac{A_1+B_1}{2}\right)$ (01) For $0 < m_1 \leq A_1, B_1 \leq M$ Lin [6] derived the following reverse form of the inequality (01)

$\phi(A_1 \# B_1) \leq \mathrm{K}(\mathrm{h})\phi\left(\frac{A_1+B_1}{2}\right)$, (02)

Where ϕ is positive unital linear map and where $K(h,2) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is known as Kantorovich constant and simple denoted by K(h).

Let$A_1, B_1 \geq 0$.Ando[1] proved the following inequality

$\Phi(A_1 \# B_1) \leq \Phi(A_1)\#\Phi(B_1)$. (03)

By inequalities (01) and (03) we have

$\phi(A_1 \# B_1) \leq \mathrm{K}(\mathrm{h}) \leq \Phi((A_1)\#\Phi(B_1))$. (04)

It is well known that positive operators $A_1, B_1 \in L(\mathcal{H})$, Lowner Heinz inequality states that for $(0 \leq p \leq 1)$.

If $A_1 \leq B_1$ then, we have ${A_1}^p \leq {B_1}^p$ (05)

In general, for $p > 1$ the inequality (05) is not true It is interesting to know that what kinds of inequalities preserve the relation when $p > 1$.

Lin in [6] proved that we can square the inequalities (02) and (04) as following

$\Phi^2\left(\frac{A_1+B_1}{2}\right) \leq K^2(h)\Phi^2(A_1 \# B_1)$, (06) $\Phi^2\left(\frac{A_1+B_1}{2}\right) \leq K^2(h)(A_1)\#\Phi(B_1)^2$. (07)

Zhang [9] generalized the inequalities (6) and (7) as follows

$$\Phi^p\left(\frac{A_1+B_1}{2}\right) \leq \frac{\left(K(h)(M_1^2+m_1^2)\right)^p}{16 M_1^p m_1^p}\Phi^p(A_1 \# B_1), \quad (08)$$

$$\phi\left(\frac{A_1+B_1}{2}\right) \leq \frac{\left(K(h)(M_1^2+m_1^2)\right)^p}{16 M_1^p m_1^p}\phi\left((A_1 \# \phi(B_1)\right)^p \quad (09)$$

where $K(h,2) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$

For $0 < m_1 \leq A_1 \leq m_2 \leq M_2 \leq B_1 \leq M_1\ and\ p \geq 1$, Moradi et al [5] improved the inequalities (06) and (07) as follows

$$\Phi^2\left(\frac{A_1+B_1}{2}\right) \leq \frac{K^2(h)}{\left(1+\frac{(\log M_2 \backslash m_2)^2}{8}\right)^2}\Phi^2(A_1 \# B_1), \quad (10)$$

$$\Phi^2\left(\frac{A_1+B_1}{2}\right) \leq \frac{K^2(h)}{\left(1+\frac{\left(\frac{\log M_2}{m_2}\right)^2}{8}\right)^2}(\Phi(A)\#\Phi(B))^2. \quad (11)$$

For $0 < m_1 \leq A_1 \leq m_2 \leq M_2 \leq B_1 \leq M_1\ and\ p \geq 1$, Yang and Lu [8] proved that obtained the generalized forms of the inequalities (10) and (11) as following

Let $0 < m \leq A_1 \leq m' \leq M' \leq B_1\ and\ p \geq 1$, we have

$\Phi^{2p}(A_1\nabla_t B_1) \le \left(\frac{K(h,2)}{4^{\frac{1}{p}-1}\left(1+Q(t)\left(log\frac{M'}{m'}\right)^2\right)}\right)^{2p} \Phi^{2p}(A_1\#_t B_1)$, (12) $\Phi^{2p}(A_1\nabla_t B_1) \le \left(\frac{K(h,2)}{4^{1\backslash p-1}\left(1+Q(t)\left(log\frac{M'}{m'}\right)^2\right)}\right)^{2p} \left(\Phi(A_1)\#_t\Phi(B_1)\right)^{2p}$ (13)

$t \in [0,1], h = \frac{M}{m}, Q(t) = \frac{t^2}{2}\left(\frac{1-2t}{t}\right)^{2t}, Q(0) = Q(1) = 1$

Let $0 < m_1 \le A_1 \le m_2 \le M_2 \le B_1 \le M_1 and\ p \ge 2$, we have

$\Phi^{2p}(A_1\nabla_u B_1) \le \left(\frac{K(h,2)(M_1^2+m_1^2)}{16M_1^{2p}m_1^{2p}\left(1+Q(u)\left(log\frac{M_2}{m_2}\right)^2\right)}\right)^{2p} \Phi^{2p}(A_1\#_u B_1)$, (14)

$\Phi^{2p}(A_1\nabla_u B_1) \le \left(\frac{K(h,2)(M_1^2+m_1^2)}{16M_1^{2p}m_1^{2p}\left(1+Q(u)\left(log\frac{M_2}{m_2}\right)^2\right)}\right)^{2p} \left(\Phi(A_1) \neq_u \Phi(B_1)\right)^{2p}$.

(15)

$t \in [0,1], h = \frac{M}{m}, Q(t) = \frac{t^2}{2}\left(\frac{1-2t}{t}\right)^{2t}, Q(0) = Q(1) = 1$

In this article in section 2 ,we shall improve the inequalities (12)-(15)

MAIN RESULTS

We will need the following important lemmas to prove our main results

Lemma 1 Let $A_1, B_1 > 0$. Then the following norm inequality holds:

$$\|A_1B_1\| \le \frac{1}{4}\|A_1 + B_1\|^2.$$

Lemma 2 Let $A_1 > 0$. Then the following result holds for every positive unital linear map Φ

$$\Phi^{-1}(A_1) \le \Phi(A_1)^{-1}$$

Lemma 3 Let $A_1, B_1 \ge 0$ and $1 \le r \le +\infty$,then the following inequality holds:

$\|{A_1}^r + {B_1}^r\| \le \|(A_1 + B_1)^r\|$.

Lemma 4 $1 + Q(t)\log\left(\frac{M_2}{m_2}\right)(A_1\#_t B_1) \le A_1^{-1}\nabla_t B_1^{-1}$

Where $Q(t) = \frac{t^2}{2}\left(\frac{1-2t}{t}\right)^{2t}, Q(0) = Q(1) = 0.$

Now we Shall prove the first main theorem of this article.

Theorem 1 Let $0 < m_1 \le A_1 \le m_2 \le M_2 \le B_1 \le M_1$ and $p \ge 1$,then for all positive unital linear map Φ,we have

$$\Phi^{2p}\left(A_1\nabla_t B_1 + M_1m_1\left({A^{-1}}_1\nabla_t B_1^{-1} - \left(1 + Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)(A_1^{-1}\#_t B_1^{-1})\right)\right) \le \left(\frac{K(h)}{4^{\frac{1}{p}-1}\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)}\right)^{2p} \Phi^{2p}(A_1\#_t B_1), \quad (16)$$

And

$$\Phi^{2p}\left(A_1\nabla_t B_1 + M_1m_1\left({A^{-1}}_1\nabla_t B_1^{-1} - \left(1 + Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)(A_1^{-1}\#_t B_1^{-1})\right)\right) \le \left(\frac{K(h)}{4^{\frac{1}{p}-1}\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)}\right)^{2p} (\Phi\,(A_1)\#_t\Phi(B_1)^{2p} \quad (17)$$

Proof: We shall apply lemma (1-4) to obtain our results

By $0 < m_1 \le A_1 \le m_2 \le M_2 \le B_1 \le M_1$, we have $(1-t)(M_1I - A_1)(m_1I - A_1)A_1^{-1} \le 0$,

This is equivalent to

$(1-t)A_1 + (1-t)M_1m_1A_1^{-1} \le (1-t)(M_1 + m_1)I$. (18)

Similarly, we have

$tB_1 + tM_1m_1B_1^{-1} \le t(M_1 + m_1)I$ (19)

By Adding (18) and (19) we obtain

$A_1\nabla_t B_1 + M_1m_1A_1\nabla_t B_1 \le (M_1 + m_1)I$ (20)

Compute

$$\left\|\begin{array}{c}\Phi^p\left(A_1\nabla_t B_1 + M_1m_1\left(A_1^{-1}\nabla_t B_1^{-1} - \left(1 + Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)A_1^{-1}\#_t B_1^{-1}\right)\right)\\ M_1^pm_1^p\left(\left(1 + Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)\right)^p \Phi^{-p}(A_1B_1)\end{array}\right\|$$

$$\le \frac{1}{4}\left\|\begin{array}{c}\Phi^p\left(A_1\nabla_t B_1 + M_1m_1\left(A_1^{-1}\nabla_t B_1^{-1} - \left(1 + Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)A_1^{-1}\#_t B_1^{-1}\right)\right)\\ + M_1^pm_1^p\left(\left(1 + Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)\right)^p \Phi^{-p}(A_1\#B_1)\end{array}\right\|^2$$

$$\leq \frac{1}{4}\left\|\begin{matrix}\Phi\left(A_1\nabla_t B_1 + M_1 m_1\left(A_1^{-1}\nabla_t B_1^{-1} - \left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)A_1^{-1}\#_t B_1^{-1}\right)\right) \\ +M_1 m_1\left(\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)\right)^p \Phi^{-1}(A_1\# B_1)\end{matrix}\right\|^{2p}$$

$$\leq \frac{1}{4}\left\|\begin{matrix}\Phi\left(A_1\nabla_t B_1 + M_1 m_1\left(A_1^{-1}\nabla_t B_1^{-1} - \left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)A_1^{-1}\#_t B_1^{-1}\right)\right) \\ +M_1 m_1\left(\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)\right)^2 \Phi((A_1\#_t B_1)^{-1})\end{matrix}\right\|^{2p}$$

$$= \frac{1}{4}\left\|\begin{matrix}\Phi\left(A_1\nabla_t B_1 + M_1 m_1\left(A_1^{-1}\nabla_t B_1^{-1} - \left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)A_1^{-1}\#_t B_1^{-1}\right)\right) \\ +M_1 m_1\left(\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)\right)^2 \Phi((A_1\#_t B_1)^{-1})\end{matrix}\right\|^{2p}$$

$= \frac{1}{4}\left\|\Phi\left(A_1\nabla_t B_1 + M_1 m_1(A_1^{-1}\nabla_t B_1^{-1})\right)\right\|^{2p}$

$\leq \frac{1}{4}(M_1+m_1)^{2p}$ (by (20))

So, we obtain

$$\left\|\begin{matrix}\Phi^p\left(A_1\nabla_t B_1 + M_1 m_1\left(A_1^{-1}\nabla_t B_1^{-1} - \left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)A_1^{-1}\#_t B_1^{-1}\right)\right) \\ \Phi^{-p}(A_1\# B_1)\end{matrix}\right\|$$

$$\leq \left(\frac{K(h)}{4^{\frac{1}{p}-1}\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)}\right)^{2p}$$

The Inequality (16) is proved.

Next, we shall prove inequality (17) by applying the lemmas (1-4)

Compute $\left\|\begin{matrix}\Phi^p\left(A_1\nabla_t B_1 + M_1 m_1\left(A_1^{-1}\nabla_t B_1^{-1} - \left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)A_1^{-1}\#_t B_1^{-1}\right)\right) \\ M_1^p m_1^p\left(\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)\right)^p\left(\Phi(A_1\#_t B_1)\right)^{-p}\end{matrix}\right\|$

$$\frac{1}{4}\left\|\begin{matrix}\Phi^p\left(A_1\nabla_t B_1 + M_1 m_1\left(A_1^{-1}\nabla_t B_1^{-1} - \left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)A_1^{-1}\#_t B_1^{-1}\right)\right) \\ + M_1^p m_1^p\left(\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)\right)^p\left(\Phi(A_1)\#_t\Phi(B_1)\right)^{-p}\end{matrix}\right\|^2$$

$$\leq \frac{1}{4}\left\|\begin{matrix}\Phi\left(A_1\nabla_t B_1 + M_1 m_1\left(A_1^{-1}\nabla_t B_1^{-1} - \left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)A_1^{-1}\#_t B_1^{-1}\right)\right) \\ +M_1 m_1\left(\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)\right)^2 \Phi^{-1}(A_1\#_t B_1)\end{matrix}\right\|^{2p} \leq \frac{1}{4}(M_1+m_1)^{2p}$$

So, we obtain

$$\left\|\begin{matrix}\Phi^p\left(A_1\nabla_t B_1 + M_1 m_1\left(A_1^{-1}\nabla_t B_1^{-1} - \left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)A_1^{-1}\#_t B_1^{-1}\right)\right) \\ \left(\Phi(A_1)\#_t\Phi(B_1)\right)^{-p}\end{matrix}\right\|$$

$$\leq \left(\frac{K(h)}{4^{\frac{1}{p}-1}\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)} \right)^{2p}.$$

Remark: By the lemma (4) it is clear that our theorem gives refinements of inequalities (12) and (13).

Theorem: 2

LET $0 < m_1 \leq A_1 \leq m_2 \leq M_2 \leq B_1 \leq M_1$ AND $p \geq 2$, THEN FOR ALL POSITIVE UNITAL LINEAR MAP Φ,WE HAVE

$$\left[\Phi^2(A_1\nabla_t B_1) + M_1^2 m_1^2 \frac{\left(\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)^2\right)}{K^2(h)} \left(\left(\frac{K(h)}{\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)}\right)^2 \Phi^{-2}(A_1\nabla_t B_1) - \Phi^{-2}(A_1\#_t B_1)\right)\right]^p \leq$$

$$\frac{\left(K(h)(M_1^2+m_1^2)\right)^{2p}}{16M_1^{2p}m_1^{2p}\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)^{2p}} \Phi^{2p}(A_1\#_t B_1), \quad (21)$$

And

$$\left[\Phi^2(A_1\nabla_t B_1) + M_1^2 m_1^2 \frac{\left(\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)^2\right)}{K^2(h)} \left(\left(\frac{K(h)}{\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)}\right)^2 \Phi^{-2}(A_1\nabla_t B_1) - \Phi^{-2}(A_1\#_t B_1)\right)\right]^p$$

$$\leq \frac{\left(K(h)(M_1^2+m_1^2)\right)^{2p}}{16M_1^{2p}m_1^{2p}\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)^{2p}} \left(\Phi(A_1)\#_t\Phi(B_1)\right)^{2p} \quad (22)$$ where $K(h) = \frac{(h+1)^2}{4h}, h = \frac{M_1}{m_1}, t \in [0,1], Q(t) = \frac{t^2}{2}\left(\frac{1-2t}{t}\right)^{2t}$

Proof For $0 < m_1 \leq A_1 \leq m_2 \leq M_2 \leq B_1 \leq M_1$, it is easy to verify that $m_1 I \leq \Phi(A\nabla_t B) \leq MI$.

So, we can obtain $m_1^2 I \leq \Phi^2(A\nabla_t B) \leq M_1^2 I$.

Therefore

$$\left(M_1^2 - \Phi^2(A\nabla_t B)\left(m_1^2 - \Phi^2(A\nabla_t B)\right)\Phi^{-2}(A\nabla_t B)\right) \leq 0$$

This is equivalent to

$M_1^2 m_1^2 \Phi^{-2}(A\nabla_t B) + \Phi^2(A\nabla_t B) \leq (M_1^2 + m_1^2)I$. (23)

Now we compute the results by applying Lemma (1) and Lemma (3)

$$\left\| \begin{matrix} \left[\Phi^2(A_1\nabla_t B_1)+M_1^2 m_1^2 \frac{\left(\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)^2\right)}{K^2(h)}\left(\frac{K^2(h)}{\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)^2}\Phi^{-2}(A_1\nabla_t B_1) \atop -\Phi^{-2}(A_1\#_t B_1)\right)\right]^{\frac{p}{2}} \\ M_1^p m_1^p \frac{\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)^p}{K^p(h)} \Phi^{-p}(A_1\#_t B_1) \end{matrix} \right\|$$

$$= \frac{1}{4} \left\|\Phi^2(A_1\nabla_t B_1) + M_1^2 m_1^2 \Phi^{-2}(A_1\nabla_t B_1)\right\|^p$$

$$\leq \frac{1}{4}(M_1^2+m_1^2)^p \text{ (by(23))}$$

This implies that

$$\left\|\left[\Phi^2(A_1\nabla_t B_1)+M_1^2 m_1^2 \frac{\left(\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)^2\right)}{K^2(h)}\left(\frac{K^2(h)}{\left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)}\Phi^{-2}(A_1\nabla_t B_1) - \Phi^{-2}(A_1\#_t B_1)\right)\right]^{\frac{p}{2}} \Phi^{-p}(A_1\# B_1)\right\|$$

$$\leq \frac{\left(K(h)(M_1^2+m_1^2)\right)^p}{4M_1^p m_1^p \left(1+Q(t)\left(log\frac{M_2}{m_2}\right)^2\right)^p}$$

Thus inequality (21) is proved

Proof of inequality (21) is same as of (22) we omit the details . The proof is completed.

Remark By the inequalities it is clear that our inequalities improved inequalities (16) and (17),it is clear that inequalities (21) and (22) gives the refinement of (14) and (15).

Conclusion and Recommendation: Operator inequalities that come into existence from operator - convex or from operator-monotone functions, or from algebraic considerations, are generally effective in wide range of settings, ranging from finite operators and matrices and elements in C^* algebra. Some matrix inequalities, however, exist for positive definite matrices. AM-GM inequality is an example of such inequality which holds for positive definite matrices and for positive invertible operators and needs future work. Singular values and unitarily invariant norms inequalities play a significant role in operators and matrices inequalities. These inequalities help us in the field of positive linear maps inequalities. Various positive linear maps operator inequalities can be proved, squared, improved and generalized.

REFERENCES

Ando, T.(1979) Concavity of certain maps on positive definite matrices and application to Hadamard products. *Linear Algebra Appl. 26: 203–241*.

Bhatia, R., Kittaneh, F. (1999) Notes on matrix arithmetic-geometric mean inequalities. *Linear Algebra. Appl, 308: 203–211*

R, Bhatia. (2007) Positive Definite matrices Princeton university press. Drury, S,W.(2012). On a question of Bhatia and Kittaneh. *Linear. Algebra. Appl*,437: 1955–1960.

Moradi, H, R., Mohsen E, M., Gumus, I, L., Naseri, R.(2018). A note on some inequalities for positive linear maps. *Linear and Multilinear Algebra*.66:7.1449-14460.

Lin, M. (2013). Squaring a reverse AM-GM inequality. *Stud. Math*, 215 (2): 187–194.

Tominaga, M. (2002). Specht's ratio in the Young's inequality. *Sci. Math. Japon*, 583-587.

Yang, C., Lu, F. (2018). Improving some operator Inequalities for Positive Linear aps. *Filomat* 32:12: 4333-4340.

Zhang, P,. (2015). More operator inequalities for positive linear maps. *Banach J. Math. Anal.* 9: 166-172.