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HADAMARD-TYPE INEQUALITIES FOR *H*-CONVEX FUNCTIONS

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ABSTRACT: In this paper, several new inequalities for differentiable and twice differentiable *h*-convex mapping are established, that are connected with the celebrated Hermite–Hadamard inequalities. Some applications for special means of real numbers are also provided.

INTRODUCTION

Let $f : I \subset \Re \to \Re$ be a convex function defined on interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality

$f \leq \int f(x)\, dx \leq$ (1)

is well known in the literature as the Hermite–Hadamard inequality.

A number of papers have been written on this inequality providing new proof, noteworthy extensions, generalizations and numerous applications. For example, the publications of *Dragomir*(2000), *Niculescue*(2004), *Sarikaya*(2010), *Hussain*(2009) and references cited there in.

In the research paper by *Varosane*(2007) a large class of non-negative functions, the so called *h*-convex functions is considered. This class contains several well known classes of functions such as non-negative convex functions, *s*-convex in the second sense, Godunova-Levin functions and P-functions. Let us repeat the definition of *h*-convex function with remark.

Definition

Let $h : J \subset \Re \to \Re$ be a non–negative function. A function $f: I \subset \Re \to \Re$ is said to be *h*-convex, or *f* belongs to the class SX (*h*, I), if *f* is non–negative and the following inequality

$f(tx + (1 - t)y) \leq h(t) f(x) + h(1 - t) f(y)$ (2)

holds for all $x, y \in I$ and $t \in (0, 1)$.

If the inequality (2) is reversed then *f* is said to be *h*-concave and we say that f belongs to the class SV(*h*, I).

Remarks:

1. If $h(t) = t$, then all the non-negative convex functions belong to the class SX (*h*, I) and all non-negative concave functions belong to the class SV(*h*, I).
2. If $h(t) =$ then Q (I) = SX (*h*, I)
3. If $h(t) = 1$ then P (I) $\subseteq$ SX (*h*, I)
4. If $h(t) = t^s$ where $s \in (0, 1)$ then K $\subseteq$ SX (*h*, I), where K represents the class of s-convex functions in second sense.

In his research paper, Sarikaya established a new Hadamard - type inequality for *h*-convex functions.

Theorem 1

Let $f \in$ S X (*h*, I), $a, b \in I$ with $a < b$ and $f \in$ L1 [*a*, *b*]. Then

$\leq \int \leq [f(a) + f(b)] \int h(t)\, dt$ *(3)*

The inequality (3) holds in reverse direction if $h \in$ SV (*h*, I) where *h* is non-negative function.

We give here definition of Beta-function also called the Euler- integral of the first kind, which will be helpful in our next discussion.

Definition

For $x, y > 0$, the function defined by $\beta(x, y) = \int t^{x-1}(1 - t)^{y-1}\, dt$ is called beta-function. The beta-function has symmetric property i.e., $\beta(x, y) = \beta(y, x)$

In our research paper of 2009 we discussed inequalities for differentiable and twice differentiable s-convex functions connecting with the H–H Inequality on the basis of the following lemmas.

Lemma 1

Let $f : I \subset \Re \to \Re$ be differentiable mapping on I^o, a, $b \in I^o$ (I^o is the interior of I) with $a < b$. If $f' \in$ L1 [*a*, *b*], then

$f \;-\; \int f(x)\, dx = \int (1 - t)\,[\,' + f'\,]\,dt$

Lemma 2

Let $f : I \subset \Re \to \Re$ be differentiable mapping on I^o, a, $b \in I^o$ with $a < b$. If $f'' \in$ L1 [*a*, *b*], then

$- \int f(x)\, dx = \int t(1 - t) f''(ta + (1 - t)b)\, dt$

The main purpose of this paper to establish new Hadamard-type inequalities for *h*-convex functions.We organize this paper as follows.

After the introduction, in section 2 some new Hermite–Hadamard type inequalities are discussed for differentiable and twice differentiable *h*-convex functions, section 3 gives us applications of the results from section 2 for special means.

MAIN RESULTS

Theorem 2

Let $f : I \to \Re$ be differentiable function on I^o, such that $f \in$ L1 [*a*, *b*], where $a < b$, $a, b \in I$. If $|f'|^q \in$ SX (*h*, I), for some $t \in [0, 1]$ and $q > 1$, then

$\int \leq \times$ + (4)

Proof.

From Lemma 1,

$\int \leq \int (1 - t)\,' +\, ' dt$ (5)

Consider

$I_1 = \int (1 - t)\,' dt$

By Holder Inequality

$I_1 \leq \int \times \int'$

Using *h*–convexity on $|f'|^q$, we have

$I_1 \leq \times$

Since $\int (1 - t)\, h(t)\, dt = \int t\, h(1 - t)\, dt$ and $\int (1 - t)\, h(t)\, dt = \int t\, h(t)\, dt$.

$I_1 \leq \times$ (6)

Analogously,

$I_2 \leq \times$ (7)

Using the inequalities (6) and (7) in (5) we get (4).

Theorem 3

Let $f : I \to \Re$, $I \subset [0, \infty)$ be differentiable function on I^o such that $f \in$ L1 [*a*, *b*], where $a, b \in I$ with $a < b$. If $|f'|^q \in$ SX (*h*, I), for some $t \in [0, 1]$ and $q > 1$ with p

= , then
∫≤ × ″ +
″∫(8)

Proof.

As in Theorem 2, consider

$I_1 = \int(1-t)$ ′

By Holder's Inequality

$I_1 \le \int \int'$

Since $|f'(x)|^q \in$ SX (h, I)

$I_1 \le \times$ ′′∫∫

as ∫= ∫$d\,t$

$I_1 \le \times$ ″∫ (9)

Analogously,

$I_2 \le \times$ ″∫ (10)

Using (9) and (10) in (5) we get (8).

Theorem 4

Let $f: I \to \Re$, $I \subset [0, \infty)$ be differentiable function on I^o such that $f \in L1\,[a, b]$, where $a, b \in I$ with $a < b$. If $|f'| \in$ SX (h, I), for some $t \in [0, 1]$ and $q > 1$, then

∫≤ ″∫
+′∫ (11)

Proof.

Proof is similar to proof of Theorem 2 and 3.

Theorem 5

Let $f: I \to \Re$, $I \subset [0, \infty)$ be differentiable function on I^o such that $f \in L1\,[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q \in$ SV (h, I), for some $t \in [0, 1]$ and $q > 1$ with $p =$, then ∫≤ × ″(12)

Proof.

Proceeding as in Theorem 2, consider

$I_1 = \int(1-t)$ ′$d\,t \le \int'$

as $|f'|^q \in$ SV (h, I) and by inequality (3),

∫′ ≤ ′

Therefore

$I_1 \le$ ′ (13)

Analogously,

$I_2 \le$ ′(14)

Using the inequalities (13) and (14) in (5) we obtain (12).

Remarks:

For $h(t) = t$ in relations (4), (8), (11) and (12) provided the lower estimate of lower classical Hadamard difference, that is the new improvements of lower Hadamard inequality.

Theorem 6

Let $f: I \subset [0, \infty) \to \Re$ be twice differentiable function on I^o such that $f'' \in L1[a, b]$, where $a, b \in I$ with $a < b$. If $|f''|^q \in$ SX (h, I), for some $t \in [0, 1]$ and $q > 1$ with = 1, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{2\times 6^{1/p}} \times$$ ″″∫

(15)

Proof.

From Lemma 2,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \int t\,(1-t)\,|f''(ta+(1-t)\,b)|\,d\,t \quad (16)$$

Consider, $I_3 = \int t\,(1-t)\,|f''(ta+(1-t)\,b)|\,d\,t$

By Holder's Inequality,

$I_3 \le \int \times \int$″

Since $|f''|^q \in$ SX (h, I), therefore

$I_3 \le \times$ ″∫∫

as ∫ = ∫

∴ $I_3 \le$ ″″× ∫ (17)

Now using the inequality (17) in (16) to get (15).

Theorem 7

Let $f: I \subset [0, \infty) \to \Re$ be twice differentiable function on I^o such that $f'' \in L1\,[a, b]$, where $a, b \in I$ with $a < b$. If $|f''|^q \in$ SX (h, I), then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{2}\,\beta^{1/p}\,[p+1, p+1] \times$$

″″∫ ... (18)

Proof.

As in Theorem 6, consider

$I_3 = \int t\,(1-t)\,|f''(ta+(1-t)\,b)|\,d\,t$

Applying Holder's Inequality, and h-convexity on $|f''|^q$, we have

$I_3 \le \beta^{1/p}\,[p+1, p+1] \times$ ″″∫ (19)

Using the inequality (19) in (16) we get (18).

Theorem 8

Let $f: I \to \Re$, $I \subset [0, \infty)$ be differentiable mapping on I^o such that $f \in L1\,[a, b]$, where $a, b \in I$ with $a < b$. If $|f''| \in$ S X (h, I), for some $t \in [0, 1]$ and $q > 1$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \times\,(\,|f''(a)|^q + |f''(b)|^q\,)\int \quad (20)$$

Proof.

Proof is similar to proof of Theorem 6 and 7.

Theorem 9

Let $f: I \to \Re$, $I \subset [0, \infty)$ be differentiable function on I^o such that $f \in L1\,[a, b]$, where $a, b \in I$ with $a < b$. If $|f''|^q \in$ SV (h, I) for some $t \in [0, 1]$ and $q \ge 1$, then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \quad \leq \ \times\ \beta^{1/p}(p+1\ ,\ p+1)\ ''$$

…. (21)

Proof.

As in Theorem 6, consider

$I_3 = \int t(1-t)\,|f''(ta+(1-t)b)|\,dt$

Applying Holder's Inequality, and *h*-concavity on $|f''|^q$ from (3)

$$I_3 \leq \beta^{1/p}(p+1, p+1)\left[2h\left(\frac{1}{2}\right)\right]^{-1/q}\ ''$$

which completes the proof.

Remarks:

For $h(t) = t$, in relations (15), (18) , (20) and (21) provide the upper estimate of upper classical Hadamard difference, that is, the new improvements of upper Hadamard inequality.

Application to some Special Means

We now consider the applications of our theorem to the special means.

(a) The arithmetic mean;

$A = A(a, b)$: = , $a, b > 0$,

(b) The geometric mean;

$G = G(a, b)$: = , $a, b > 0$,

(c) The Harmonic mean:

$H = H(a, b)$: = , $a, b > 0$,

(d) The logarithmic mean:

$L = L(a, b)$: = $\neq$, $a, b > 0$

(e) The identric mean:

$I = I(a, b)$: = $\neq$, $a, b > 0$

(f) The *p*- logarithmic mean:

$L_p = L_p(a, b)$: = $\neq$, $p \in \Re \setminus \{-1, 0\}$: $a, b > 0$

The following proposition holds:

Proposition 1

Let $a, b \in \Re$, $0 \notin [a, b]$ and $n \in Z$ and $|n| \geq 2$. Then, for all $q > 1$, we have

$\leq \times [(|a|^{q(n-1)} + 2A^{q(n-1)}(a, b)) + (|b|^{q(n+1)} + 2A^{q(n-1)}(a, b))]$

Proof.

The proof is immediate from Theorem 2 applied for $f(x) = x^n$, $x \in \Re$, $n \in Z$ and $|n| \geq 2$.

Proposition 2

Let $a, b \in \Re$, then for $a < b$, we have

$|G(a, b) - L(a, b)| \leq L^{-2}(a, b) \times [A(a, b) + 2G(a, b)]$

Proof.

The assertion follows Theorem 6 applies for $f(x) = e^x$, $x \in \Re$.

Proposition 3

Let $a, b \in [0, \infty)$ and $a < b$. Then, for all $q > 1$. We have

$ln \leq (b-a)^2 \times \beta^{1/p}[p+1, p+1]\, A^{-2}(a, b)$

Proof.

The assertion follows from Theorem 9 applied to $f: [0, \infty) \rightarrow [0, \infty)$, $f(x) = ln(x+1)$ and details are omitted.

Proposition 4

If $a, b \in \Re$, $0 \notin [a, b]$, then for all $q > 1$, the following inequality holds:

$|H^{-1}(a, b) - L^{-1}(a, b)| \leq \times A^{1/q}(|a|^{-3q}, |b|^{-3q})$

Proof.

The proof is obvious from Theorem 4 applied for $f(x) = \frac{1}{x}$, $x \in [a, b]$

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